Testimony

This is to certify that the candidate **Rania Saeed Al-Ghamdi** worked under the supervision of the undersigned and completed this thesis to meet the partial requirement of a degree of Master of science in mathematics.

Dr. Noura Omair Al-Shehri.

(Supervisor)

ABSTRACT

The subject of this thesis depends on the study of the concept of modern algebraic concepts, a concept of K-algebras where this concept was originated for the first time in 2005 by K. H. Dar and M. Akram [22] along with two other concepts: soft sets theory and fuzzy soft sets theory.

In chapter 1: The previous concepts of K-algebras, soft sets theory, fuzzy soft theory were introduced. Also, some of the properties, theories and results were reviewed in order to take advantage of them in the coming chapters of the thesis.

In chapter 2: Soft set theory was applied to K-algebras and some examples were introduced. The notion of abelian soft K-algebras was presented. Also the concept of soft intersection K-subalgebras was discussed and some of the properties of the above three concepts were invistigated. These concepts and results were submitted as an artical and was published successfuly [16].

In chapter 3: The concept of fuzzy soft K-subalgebras was introduced and some of their properties were investigated. Fuzzy soft images and fuzzy soft inverse images of fuzzy soft K-subalgebras were discussed. The notion of an $(\in, \in \vee q)$-fuzzy soft K-subalgebra which is a generalization of a fuzzy soft K-subalgebra was defined. Also the notion of $(\in_\alpha, \in_\alpha \vee q_\beta)$-fuzzy soft K-subalgebras was presented and some of their properties were described. These concepts also were published as an artical in a scintific journal [8].

ABSTRACT(Arabic)

Acknowledgement

First and foremost, I thank God Almighty for his generosity that reconciled me to complete this thesis. Then I extend my sincere thanks to my supervisor Dr. Noura Al-Shehri to the great effort, direction and follow up me at all stages of the thesis from the beginning untill reaching the stage of completion and submission.

Also, I thank my parents who stood behind me and supported me by prayers and guidance.

Finally I submit my thanks and gratitude to my husband who gave of his time, encouragement and support.

Dedication

To whom Allah almighty crowned him by dignity and respect, who taught me without waiting feedbacks and to whom I carry his name with all proudness .. my dear father.

To the meaning of love and meaning of sympathy and to whom her prayers were the secret of my success .. my darling mother.

To the partner of my life and the sharer my successes, who encouraged, supported and stood behind me in my way to success and excellence .. my beloved husband.

To the clean soft hearts and innocent souls, to the flowers of my lif .. my daughters (Lyan and Lena).

I dedicate this effort which took its time and gave its result. A result which I dreamed to achieve and I achieved with the help and guidance of Allah Almighty.

Rania S. Al-Ghamdi

Contents

PREFACE

A soft set theory as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches was proposed by Molodtsov in 1999 [40]. He pointed out several directions for the applications of soft sets. In 2002, Maji et al. [38] described the application of soft set theory to a decision making problem. He [37] also studied several operations on the theory of soft sets. Few years later, Chen et al. [20] presented a new definition of soft set parametrization reduction, and compared this definition to the related concept of attributes reduction in rough set theory. The works on the soft set theory are progressing rapidly. The algebraic structure of set theories dealing with uncertainties has been studied by some authors. The notion of a K-algebra $(G, ., \odot, e)$ was introduced by Dar and Akram [22]. A K-algebra is an algebra built on a group $(G, ., e)$ by adjoining an induced binary operation $\odot$ on G which is attached to an abstract K-algebra $(G, ., \odot, e)$. This system is, in general non-commutative and non-associative with a right identity e, if $(G, ., e)$ is non-commutative. Dar and Akram further renamed a K-algebra on a group G as a K(G)-algebra [21] due to its structural basis G. The $K(G)$-algebra has been characterized by using its left and right mappings in [21]. Recently, Dar and Akram [23] have further proved that the class of $K(G)$-algebras is a generalized class of B-algebras [41] when $(G, ., e)$ is a non-abelian group, and they also proved that the $K(G)$-algebra is a generalized class of the class of $BCH/BCI/BCK$-algebras [32] when $(G, ., e)$ is an abelian group.

The most appropriate theory for dealing with vagueness is the theory of fuzzy sets developed by Zadeh [47]. Since then it has become a vigorous area of research in different domains such as engineering, medical science, social science, physics, statistics, graph theory, artificial intelligence, signal processing, multiagent systems, pattern recognition, robotics, computer networks, expert systems, decision making, automata theory. The authors applied fuzzy set theory to BCK-algebras, B-algebras, MTL-algebras, hemirings, implicative algebras, lattice implication algebras and incline algebras. In [4] Akram and Alshehri applied fuzzy set theory to K-algebras, they introduced notion of fuzzy K-ideals of K-algebras and investigated some of their properties. They characterized ascending and descending chains of K-ideals by the corresponding fuzzy K-ideals and discussed some properties of characteristic fuzzy K-ideals of K-algebras. They also constructed a quotient K-algebra via fuzzy K-ideal and presented the fuzzy isomorphism theorems.

In our study we investigate applications of soft set theory in K-algebras, we introduce a soft K-algebras and establish some related properties. Also, we apply the notion of fuzzy soft sets to the theory of K-algebras by introducing the notion of fuzzy soft K-algebras and deriving their basic properties.

Chapter 1

Fundamental Concepts

1.1 On a K-algebra Built on a Group

This chapter is devoted to collect some basic notions and important terminology with a view to making our thesis assey contained as possible.

The notion of K-algebra was first introduced by Dar and Akram [22] in 2003 and published in 2005. A K-algebra is an algebra built on a group by adjoining an induced binary operation on group which is attached to an abstract K-algebra. This system is, in general non-commutative and non-associative with a right identity e, if group is non-commutative.

Definition 1.1.1 [22] Let $(G, ., e)$ be a group with the identity e such that $x^2 \neq e$ for some $x(\neq e) \in G$. Then a K-algebra is a structure $K = (G, ., \odot, e)$ on a group G in which induced binary operation $\odot : G \times G \to G$ is defined by $\odot(x, y) = x \odot y = xy^{-1}$ and satisfies the following axioms:
(K1) $(x \odot y) \odot (x \odot z) = (x \odot ((e \odot z) \odot (e \odot y))) \odot x$,
(K2) $x \odot (x \odot y) = (x \odot (e \odot y)) \odot x$,

(K3) $x \odot x = e$,

(K4) $x \odot e = x$,

(K5) $e \odot x = x^{-1}$

for all $x, y, z \in G$.

In what follows, we denote a K-algebra by K unless otherwise specified.

Definition 1.1.2 [24] A K-algebra K is called abelian if and only if $x \odot (e \odot y) = y \odot (e \odot x)$ for all $x, y \in G$.

If K is abelian, then the axioms $(K1)$ and $(K2)$ can be written as:

$(\overline{K1})$ $(x \odot y) \odot (x \odot z) = z \odot y$.

$(\overline{K2})$ $x \odot (x \odot y) = y$.

Example 1.1.3 [22] Let $S_3 = \{e, a, b, x, y, z\}$ be the symmtric group where $e = (1), a = (123), b = (132), x = (12), y = (13), z = (23)$, define the operation $\odot$ by the following Cayley's table:

$\odot$	e	x	y	z	a	b
e	e	x	y	z	b	a
x	x	e	b	a	y	z
y	y	a	e	b	z	x
z	z	b	a	e	x	y
a	a	y	z	x	e	b
b	b	z	x	y	a	e

Then $(S_3, ., \odot, e)$ is a K-algebra.

Example 1.1.4 [7] Let $S = V_2(R) = \{(x, y) : x, y \in \mathbb{R}\}$ be the set of all 2-dimensional real vectors which forms an additive $(+)$ abelian group. Define the operation $\odot$ on S by $x \odot y = x - y$ for all $x, y \in G$. Then $(S, +, \odot, e)$ is a K-algebra.

Definition 1.1.5 [22] A nonempty subset H of a K-algebra $K = (G, ., \odot, e)$ is called K-subalgebra if:

(i) $e \in H$,
(ii) $h_1 \odot h_2 \in H$ for all $h_1, h_2 \in H$.

Note that every subalgebra of K contains the identity e of the group $(G, ., e)$.

Definition 1.1.6 [3] The K-algebra called improper since the group G is elementary abelian 2-group, i.e., $x \odot y = x.y^{-1} = x.y$, and called proper if G is not an elementary abelian 2-group.

Example 1.1.7 [22] Consider the K-algebra K which is given in example 1.1.3. We can check that K-subalgebra $H_1 = (A_3, ., \odot. e)$ is a proper subalgebra having the following table:

$\odot$	e	a	b
e	e	b	a
a	a	e	b
b	b	a	e

Also $H_2 = \{e, x\}$ is an improper K-subalgebra having the cayley's table:

$\odot$	e	x
e	e	x
x	x	e

In 2007, Dar and Akram [24] discussed some properties of K-subalgebras and K-ideals of K-algebras [10]. Also, they introduced the notion of K-homomorphisms of K-algebras.

Definition 1.1.8 [10] Let A be a nonempty subset of K, then A is called an ideal of K if it satisfies the following conditions:
(i) $e \in A$,
(ii) $x \odot y \in A$, $y \odot (y \odot x) \in A \Longrightarrow x \in A$ for all $x, y \in G$

Definition 1.1.9 [3] A nonempty subset I of K is called a K-ideal of K if it satisfies the following conditions:

(i) $e \in I$,
(ii) $x \odot (y \odot z) \in I$, $y \odot (y \odot x) \in I \Longrightarrow x \odot z \in I$
for all $x, y, z \in G$.

Proposition 1.1.10 [3] (1) Every K-ideal is an ideal.
(2) Any ideal of a K-algebra is a subalgebra of K.

Example 1.1.11 [24] Consider the K-algebra $(A_3, ., \odot, e)$ which is given in example 1.1.6., it easy to check that $(A_3, ., \odot, e)$ is an ideal of K-algebra $K = (S_3, ., \odot, e)$.

Proposition 1.1.12 [24] If I_1 and I_2 are two K-subalgebras (ideals) of K then:
(a) $I_1 \cap I_2$ is a K-subalgebra (ideal) of K.
(b) $I_1 \odot I_2 = \{x_1 \odot x_2 \text{ where } x_1 \in I_1, x_2 \in I_2\}$ is a K-subalgebra (ideal) of K if and only if $I_1 \odot I_2 = I_2 \odot I_1$ (either I_1 or I_2 is ideal).

Definition 1.1.13 [24] Suppose $K_1 = (G_1, ., \odot, e_1)$ and $K_2 = (G_2, ., \odot, e_2)$ are two K-algebras. A mapping $\varphi : K_1 \to K_2$ is called a K-homomorphism from K_1 into K_2 if $\varphi(x \odot y) = \varphi(x) \odot \varphi(y)$ for all $x, y \in K_1$.

Definition 1.1.14 [24] Let $K_1 = (G_1, ., \odot, e_1)$ and $K_2 = (G_2, ., \odot, e_2)$ be two K-algebras and φ be a K-homomorphism from K_1 into K_2. The subset $Ker\varphi = \{x \in K_1 : \varphi(x) = e_2\}$ of K_1 is called the kernel of φ.

Proposition 1.1.15 [24] Let $K_1 = (G_1, ., \odot, e_1)$ and $K_2 = (G_2, ., \odot, e_2)$ be two K-algebras and $\varphi \in Hom(K_1, K_2)$. Then, for $x_1, y_1 \in K_1$ and $\varphi(x_1), \varphi(y_1) \in K_2$, we conclude that:
(1) $\varphi(e_1) = e_2$.
(2) $\varphi(x_1) = \varphi(x_1^{-1})$.
(3) $\varphi(e_1 \odot x_1) = e_2 \odot \varphi(x_1)$.
(4) $\varphi(x_1 \odot x_2) = e_2$, if and only if $\varphi(x_1) = \varphi(x_2)$.
(5) If H_1 is a subalgebra of K_1 then $\varphi(H_1)$ is a subalgebra of K_2.
(6) If H_1 is an ideal of K_1 then $\varphi(H_1)$ is an ideal of K_2.

Next, we recall some properties of K-algebras

Proposition 1.1.16 [25] In K-algebras K the following statements are equivalent:

(a) A K-algebra K is abelian,

(b) $x \odot (x \odot y) = y$,

(c) $(x \odot y) \odot z = (x \odot z) \odot y$,

(d) $(e \odot x) \odot (e \odot y) = e \odot (x \odot y)$,

(e) $(x \odot y) \odot (x \odot z) = z \odot y$

for all $x, y, z \in G$.

Proposition 1.1.17 [25] If the class of K-algebras K is an abelian. Then the following identities hold for all $x, y, z \in G$:

(a) $x \odot (e \odot y) = y \odot (e \odot x)$,

(b) $(x \odot y) \odot z = (x \odot z) \odot y$,

(c) $(x \odot (x \odot y)) \odot y = e$,

(d) $e \odot (x \odot y) = (e \odot x) \odot (e \odot y) = y \odot x$.

Proposition 1.1.18 [25] In an abelian K-algebra K the following assertions are equivalent:

(a) $x \odot (y \odot z)$,

(b) $(x \odot y) \odot (e \odot z)$,

(c) $z \odot (y \odot x)$

Proposition 1.1.19 [25] Let K be a K-algebra on non-abelian group G. Then the following identities hold in K for all $x, y, z \in G$:

(a) $x \odot (y \odot z) = (x \odot (e \odot z)) \odot y$,

(b) $(x \odot y) \odot z = x \odot (z \odot (e \odot y))$,

(c) $e \odot (x \odot y) = y \odot x$,

(d) $e \odot (e \odot x) = x$,

(e) $x \odot (x \odot (e \odot x)) = e \odot x$,

(f) $x \odot (z \odot (e \odot x)) = (e \odot x) \odot (z \odot x) = e \odot z$,

(g) $(x \odot y) \odot (z \odot y) = x \odot z$,

(h) $(x \odot y) \odot (e \odot y) = x$,

(i) $x \odot y = e = y \odot x \Longrightarrow x = y$.

1.2 Characterization of Fuzzy K-algebras

The notion of a fuzzy subset of a set was first introduced by Zadeh [47] in 1965 as a method of representing uncertainty. Since then, fuzzy set theory has been devloped in many directions by many scholars.

Definition 1.2.1 [47] Let X be a nonempty set. A fuzzy subset μ of X is defined as a mapping from X into $[0,1]$, where $[0,1]$ is the usual interval of real numbers. We denote by $F(X)$ the set of all fuzzy subsets of X.

Definition 1.2.2 [47] Let μ and ν are fuzzy subsets of a set X. We say that μ is a subset of ν denoted by $\mu \subseteq \nu$ if $\mu(x) \leq \nu(x), \forall x \in X$.
If $\mu(x) = \nu(x), \forall x \in X$, then μ and ν are said to be equal and we write $\mu = \nu$.

Definition 1.2.3 [47] Let μ be a fuzzy subset of a set X and let $t \in [0,1]$. The set $\mu_t = \{x \in X : \mu(x) \geq t\}$ is called a level subset of μ

Definition 1.2.4 [47] The complement of a fuzzy subset μ is defined as $(\rceil\mu)(x) = 1 - \mu(x)$, for all $x \in X$.

Definition 1.2.5 [47] The intersection of two fuzzy subsets μ and ν of a set X is defined as $(\mu \cap \nu)(x) = \min\{\mu(x), \nu(x)\} = \mu(x) \wedge \nu(x)$, for all $x \in X$.

Definition 1.2.6 [47] The union of two fuzzy subsets μ and ν of a set X is defined as $(\mu \cup \nu)(x) = \max\{\mu(x), \nu(x)\} = \mu(x) \vee \nu(x)$, for all $x \in X$.

Definition 1.2.7 More generally, the union and intersection of any family $\{\mu_i : i \in \Omega\}$ of fuzzy subsets of a set X are defined by

$$(\cup_{i\in\Omega}\mu_i)(x) = \sup_{i\in\Omega} \mu_i(x), \forall x \in X.$$

$$(\cap_{i\in\Omega}\mu_i)(x) = \inf_{i\in\Omega} \mu_i(x), \forall x \in X.$$

Definition 1.2.8 Let μ and ν be any two fuzzy subsets of X. Then the product $\mu \circ \nu$ defined by

$$(\mu \circ \nu)(z) = \begin{cases} \vee_{z=x.y}(\mu(x) \wedge \nu(y)) & \text{if there exist } x, y \in X \text{ such that } z = x.y, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1.2.9 [42] A fuzzy subset μ of X of the form

$$\mu(y) = \begin{cases} t \in (0,1] & \text{if} \quad y = x, \\ 0 & \text{if} \quad y \neq x. \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t.

The fuzzy structures of K-algebras was introduced in [10]. Then, the notion of K-ideals of K-algebras was introduced in [4].

Definition 1.2.10 [22] A fuzzy subset μ in K is called a fuzzy K-subalgebra of K if it satisfies:

(i) $(\forall x \in G)(\mu(e) \geq \mu(x))$,

(ii) $(\forall x, y \in G)(\mu(x \odot y) \geq \min\{\mu(x), \mu(y)\})$.

Definition 1.2.11 [10] A fuzzy ideal μ of K is a mapping $\mu : G \to [0,1]$ such that

(i) $(\forall x \in G)(\mu(e) \geq \mu(x))$,

(ii) $(\forall x, y \in G)(\mu(x) \geq \min\{\mu(x \odot y), \mu(y \odot (y \odot x))\})$.

Example 1.2.12 [5] Let $K = (G, ., \odot, e)$ be a K-algebra on the cyclic group $G = \{0, a, b, c, d, f\}$ where $0 = e, a = a, b = a^2, c = a^3, d = a^4, f = a^5$ and $\odot$ is given by the following Cayley's table:

$\odot$	0	a	b	c	d	f
0	0	f	d	c	b	a
a	a	0	f	d	c	b
b	b	a	0	f	d	c
c	c	b	a	0	f	d
d	d	c	b	a	0	f
f	f	d	c	b	a	0

Let μ be a fuzzy subset in G defined by $\mu(e) = t_1$ and $\mu(x) = t_2$ for all $x \neq 0$ in G, where $t_1, t_2 \in [0, 1]$ and $t_1 > t_2$. Then it is easy to check that μ is a fuzzy ideal of K.

Definition 1.2.13 [4] A fuzzy subset μ in K is called a fuzzy K-ideal of K if it satisfies:

(i) $(\forall x \in G)(\mu(e) \geq \mu(x))$,

(ii) $(\forall x; y, z \in G)(\mu(x \odot z) \geq \min\{\mu(x \odot (y \odot z)), \mu(y \odot (y \odot x))\})$.

Example 1.2.14 [4] Consider the K-algebra $K = (G, ., \odot, e)$ on the Dihedral group $G = \{e, a, u, v, b, x, y, z\}$ where $u = a^2, v = a^3, x = ab, y = a^2b, z = a^3b$, and $\odot$ is given by the following Cayley's table:

$\odot$	e	a	u	v	b	x	y	z
e	e	v	u	a	b	x	y	z
a	a	e	v	u	x	y	z	b
u	u	a	e	v	y	z	b	x
v	v	u	a	e	z	b	x	y
b	b	x	y	z	e	v	u	a
x	x	y	z	b	a	e	v	u
y	y	z	b	x	u	a	e	v
z	z	b	x	y	v	u	a	e

Let μ be a fuzzy subset in K defined by $\mu(e) = 0.8, \mu(t) = 0.06$ for all $t \neq e$. Then μ is a fuzzy K-ideal of K.

Proposition 1.2.15 [4] Every fuzzy K-ideal of K is a fuzzy ideal of K.

Proposition 1.2.16 [10] Let μ be a fuzzy subset in K. Then μ is a fuzzy ideal of K if and only if the set $U(\mu;t)=\{x\in G:\mu(x)\geq t\}$, $t\in[0,1]$, is an ideal of K when it is nonempty.

Proposition 1.2.17 [5] Let μ be a fuzzy ideal of K and let $x\in K$. Then $\mu(x)=t$ if and only if $x\in U(\mu;t)$ and $x\notin U(\mu;s)$ for all $s>t$.

For a fuzzy point x_t and a fuzzy set μ in a set X, Pu and Liu [42] gave meaning to the symbol $x_t\alpha\mu$, where $\alpha\in\{\in,q,\in\vee q,\in\wedge q\}$. A fuzzy point x_t is called belong to a fuzzy set μ , written as $x_t\in\mu$, if $\mu(x)\geq t$. A fuzzy point x_t is said to be quasicoincident with a fuzzy set μ , written as $x_tq\mu$, if $\mu(x)+t>1$. To say that $x_t\in\vee q\mu$ (resp. $x_t\in\wedge q\mu$) means that $x_t\in\mu$ or $x_tq\mu$ (resp. $x_t\in\mu$ and $x_tq\mu$).
$x_t\bar{\alpha}\mu$ means that $x_t\alpha\mu$ does not hold, where $\alpha\in\{\in,q,\in\vee q,\in\wedge q\}$.

Definition 1.2.18 [9] A fuzzy subset μ in K is called an $(\in,\in\vee q)$-fuzzy K-subalgebra of K if it satisfies the following conditions:
(1) $x_s\in\mu\rightarrow e_s\in\vee q\mu$,
(2) $x_s\in\mu,y_t\in\mu\rightarrow(x\odot y)_{\min\{s,t\}}\in\vee q\mu$.
for all $x,y\in G$, $s,t\in(0,1]$.

Proposition 1.2.19 [9] Let K be a K-algebra. A fuzzy subset μ in K is a fuzzy K-subalgebra of K if and only if the following assertion is valid.

$$x_t\in\mu,y_s\in\mu\ \rightarrow(x\odot y)_{\min\{s,t\}}\in\mu\ for\ all\ x,y\in G,s,t\in(0,1].$$

1.3 Main Notions of Soft Set Theory

In 1999, Molodtsov [40] initiated soft set theory as a new approach for modelling uncertainties. A soft set can be determined by a set-valued mapping assigning to each parameter exactly one crisp subset of the universe. More specifically, we can define the notion of soft set in the following way.

Definition 1.3.1 Let U be an initial universe and E be a set of parameters. Let $P(U)$ denote the power set of U and let A be a nonempty subset of E. A pair $F_A = (F, A)$ is called a soft set over U, where $A \subseteq E$ and $F : A \to P(U)$ is a set-valued mapping, called the approximate function of the soft set F_A. It is easy to represent a soft set F_A by a set of ordered pairs as follows:

$$F_A = (F, A) = \{(x, F(x)) : x \in A\}$$

Example 1.3.2 [37] Suppose that U is the set of houses under consideration and E is the set of parameters. Each parameter is a word or a sentence.
$E = \{$expensive, beautiful, wooden, cheap, in the green surroundings, modern, in good repair, in bad repair$\}$.
In this case, to define a soft set means to point out expensive houses, beautiful houses, and so on. The soft set F_E describes the "attractiveness of the houses" which Mr. X (say) is going to buy.
We consider below the same example in more detail for our next discussion.
Suppose that there are six houses in the universe U given by

$$U = \{h_1, h_2, h_3, h_4, h_5, h_6\} \text{ and } E = \{e_1, e_2, e_3, e_4, e_5\}$$

where
e_1 stands for the parameter "expensive",

e_2 stands for the parameter "beautiful",
e_3 stands for the parameter "wooden",
e_4 stands for the parameter "cheap",
e_5 stands for the parameter "in the green surroundings".
Suppose that

$$F(e_1) = \{h_2, h_4\},$$

$$F(e_2) = \{h_1,\ h_3\},$$

$$F(e_3) = \{h_3,\ h_4,\ h_5\},$$

$$F(e_4) = \{h_1, h_3, h_5\},$$

$$F(e_5) = \{h_1\}.$$

The soft set F_E is a parametrized family $\{F(e_i), i = 1, 2, 3, ., 5\}$ of subsets of the set U and gives us a collection of approximate descriptions of an object. Consider the mapping F which is "houses (.)" where dot (.) is to be filled up by a parameter $e \in E$. Therefore, $F(e_1)$ means "houses (expensive)" whose functional-value is the set $\{h_2, h_4\}$.

Thus, we can view the soft set F_E as a collection of approximations as below:

$F_E = \{$expensive houses$= \{h_2, h_4\}$, beautiful houses$= \{h_1, h_3\}$, wooden houses$= \{h_3, h_4, h_5\}$, cheap houses $= \{h_1, h_3, h_5\}$, in the green surroundings $= \{h_1\}\}$, where each approximation has two parts:
(i) a predicate p, and
(ii) an approximate value-set v (or simply to be called value-set v).
For example, for the approximation "expensive houses $= \{h_2, h_4\}$", we have the following:
(i) the predicate name is expensive houses; and
(ii) the approximate value set or value set is $\{h_2, h_4\}$.

U	Expensive	Beautiful	Wooden	Cheep	In the green surroundings
h_1	0	1	0	1	1
h_2	1	0	0	0	0
h_3	0	1	1	1	0
h_4	1	0	1	0	0
h_5	0	0	1	1	0
h_6	0	0	0	0	0

Thus, a soft set, F_E can be viewed as a collection of approximations below:

$$F_E = \{p_1 = v_1, p_2 = v_2, ..., p_n = v_n\}.$$

For the purpose of storing a soft set in a computer, we could represent a soft set in the form of above table, (corresponding to the soft set in the above example).

In 2003, Maji et al. [37] studied the theory of soft sets initiated by Molodtsov. The authors defined equality of two soft sets, subset, complement of a soft set, null soft set and absolute soft set with examples. Soft binary operations like AND, OR and also the operations of union, intersection were defined. DeMorgan's laws were verified in soft set theory.

Definition 1.3.3 [37] A soft set F_A over U is said to be a null soft set denoted by Φ, if $\forall \varepsilon \in A, F(\varepsilon) = \emptyset$.

Example 1.3.4 [37] Suppose that U is the set of wooden houses under consideration and A is the set of parameters.
Let there be five houses in the universe U given by

$$U = \{h_1, h_2, h_3, h_4, h_5\} \text{ and } A = \{\text{brick, muddy, steel, stone}\}.$$

The soft set F_A describes the "construction of the houses". The soft sets F_A is defined as

F (brick) means the brick built houses,
F (muddy) means the muddy houses,
F (steel) means the steel built houses,
F (stone) means the stone built houses.
The soft set F_A is the collection of approximations as below:
F_A = {brick built houses= $\emptyset$, muddy houses = $\emptyset$, steel built houses = $\emptyset$, stone built houses = $\emptyset$}
Therefore, F_A is null soft set.

Definition 1.3.5 [37] A soft set F_A over U is said to be an absolute soft set denoted by $\tilde{A}$, if $\forall \varepsilon \in A, F(\varepsilon) = U$.

Example 1.3.6 [37] Suppose that U is the set of wooden houses under consideration and B is the set of parameters.
Let there be five houses in the universe U given by

$$U = \{h_1, h_2, h_3, h_4, h_5\} \text{ and } B = \{\text{not brick, not muddy, not steel, not stone}\}.$$

The soft set G_B describes the "construction of the houses". The soft sets G_B is defined as
G (not brick) means the houses not built by brick,
G (not muddy) means the not muddy houses,
G (not steel) means the houses not built by steel,
G (not stone) means the houses not built by stone.
The soft set G_B is the collection of approximations as below:
G_B = {not brick built houses= $\{h_1, h_2, h_3, h_4, h_5\}$, not muddy houses = $\{h_1, h_2, h_3, h_4, h_5\}$, not steel built houses = $\{h_1, h_2, h_3, h_4, h_5\}$, not stone built houses = $\{h_1, h_2, h_3, h_4, h_5\}$}
Therefore, G_B is the absolute soft set.

Definition 1.3.7 [37] Let F_A and G_B be two soft sets over a common universe U. F_A is a said to be a soft subset of G_B, denoted by $F_A \tilde{\subseteq} G_B$, if:

(i) $A \subset B$,
(ii) $\forall \varepsilon \in A$, $F(\varepsilon)$ and $G(\varepsilon)$ are identical approximations.

Definition 1.3.8 [37] Two soft sets F_A and G_B over a common universe U are said to be soft equal if F_A is a soft subset of G_B and G_B is a soft subset of F_A. We write $F_A = G_B$.

Example 1.3.9 [37] Let $A = \{e_1, e_3, e_5\} \subset E$, and $B = \{e_1, e_2, e_3, e_5\} \subset E$. Clearly, $A \subset B$.
Let F_A and G_B be two soft sets over the same universe $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ such that
$G(e_1) = \{h_2, h_4\}, G(e_2) = \{h_1, h_3\}, G(e_3) = \{h_3, h_4, h_5\}, G(e_5) = \{h_1\}$ and $F(e_1) = \{h_2, h_4\}, F(e_3) = \{h_3, h_4, h_5\}, F(e_5) = \{h_1\}$.

Therefore, $F_A \tilde{\subset} G_B$.

Definition 1.3.10 [37] The complement of a soft set F_A is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, \rceil A) = F^c_{\rceil A}$, where $F^c : \rceil A \to P(U)$ is a mapping given by $F^c(\alpha) = U - F(\rceil \alpha), \forall \rceil \alpha \in \rceil A$.

Example 1.3.11 [37] Consider Example 1.3.2. Here F^c_E = {not expensive houses $= \{h_1, h_3, h_5, h_6\}$, not beautiful houses$= \{h_2, h_4, h_5, h_6\}$, not wooden houses$= \{h_1, h_2, h_6\}$, not cheap houses$= \{h_2, h_4, h_6\}$, not in the green surroundings houses$= \{h_2, h_3, h_4, h_5, h_6\}$}.

Definition 1.3.12 [37] Let F_A and G_B be two soft sets over a common universe U, then "F_A AND G_B" denoted by $F_A \tilde{\wedge} G_B$ is defined by $F_A \tilde{\wedge} G_B = H_{A \times B}$, where $H(x, y) = F(x) \cap G(y)$ for all $(x, y) \in A \times B$.

Example 1.3.13 [37] Consider the soft set F_A which describes the "cost of the houses" and the soft set G_B which describes the "attractiveness of the houses".
Suppose that $U = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9, h_{10}\}$, A = {very costly, costly, cheap} and B = {beautiful, in the green surroundings, cheap}.

Let
$F(\text{very costly}) = \{h_2, h_4, h_7, h_8\}$,
$F(\text{costly}) = \{h_1, h_3, h_5\}$,
$F\ (\text{cheap}) = \{h_6, h_9, h_{10}\}$,
and
$G(\text{beautiful}) = \{h_2, h_3, h_7\}$,
$G(\text{in the green surroundings}) = \{h_5, h_6, h_8\}$,
$G(\text{cheap}) = \{h_6, h_9, h_{10}\}$.
Then $F_A \tilde{\wedge} G_B = H_{A\times B}$, where
$H(\text{very costly, beautiful}) = \{h_2, h_7\}$,
$H(\text{very costly, in the green surroundings}) = \{h_8\}$,
$H(\text{very costly, cheap}) = \emptyset$,
$H(\text{costly, beautiful}) = \{h_3\}$,
$H(\text{costly, in the green surroundings}) = \{h_5\}$,
$H(\text{costly, cheap}) = \emptyset$,
$H(\text{cheap, beautiful}) = \emptyset$,
$H(\text{cheap, in the green surroundings}) = \{h_6\}$,
$H(\text{cheap. cheap}) = \{h_6, h_9, h_{10}\}$.

Definition 1.3.14 [37] Let F_A and G_B be two soft sets over a common universe U, then "F_A OR G_B" denoted by $F_A \tilde{\vee} G_B$ is defined by $F_A \tilde{\vee} G_B = H_{A\times B}$, where $H(x, y) = F(x) \cup G(y)$ for all $(x, y) \in A \times B$.

Example 1.3.15 [37] Consider the soft sets $F_A$and G_B over a common universe U which is given in Example 1.3.13. We see that $F_A \tilde{\vee} G_B = H_{A\times B}$, where
$H(\text{very costly, beautiful}) = \{h_2, h_3, h_4, h_7, h_8\}$,
$H(\text{ veIry costly, in the green surroundings}) = \{h_2, h_3, h_4, h_5, h_6, h_7, h_8\}$,
$H(\text{ very costly, cheap}) = \{h_2, h_4, h_6, h_7, h_8, h_9, h_{10}\}$,
$H(\text{costly, beautiful}) = \{h_1, h_2, h_3, h_5, h_7\}$,
$H(\text{costly, in the green surroundings}) = \{h_1, h_3, h_5, h_6, h_8\}$,

H(costly, cheap) $= \{h_1, h_3, h_5, h_6, h_9, h_{10}\}$,
H(cheap, beautiful) $= \{h_2, h_3, h_6, h_7, h_9, h_{10}\}$,
H(cheap, in the green surroundings) $= \{h_5, h_6, h_8, h_9, h_{10}\}$,
H(cheap, cheap) $= \{h_6, h_9, h_{10}\}$.

Definition 1.3.16 [37] The intersection of two soft sets F_A and G_B over a common universe U is the soft set H_C, where $C = A \cap B$ and for all $x \in C$, $H(x) = F(x)$ or $G(x)$, (as both are same set). We write $F_A \tilde{\cap} G_B = H_C$.

Example 1.3.17 [37] In example 1.3.13, intersection of two soft sets F_A and G_B is the soft set H_C, where $C = \{\text{cheap}\}$ and H(cheap) $= \{h_6, h_9, h_{10}\}$.

Definition 1.3.18 [37] The union of two soft sets F_A and G_B over a common universe U is the soft set H_C, where $C = A \cup B$ and for all $x \in C$,

$$H(x) = \begin{cases} F(x) & \text{if } x \in A - B, \\ G(x) & \text{if } x \in B - A, \\ F(x) \cup G(x) & \text{if } x \in A \cap B. \end{cases}$$

We write $F_A \tilde{\cup} G_B = H_C$.

Example 1.3.19 [37] In example 1.3.13, union of two soft sets F_A and G_B is the soft set H_C, where $C = \{$very costly, costly, cheap, beautiful, in the green surroundings$\}$ and H (very costly) $= \{h_2, h_4, h_7, h_8\}$, H (costly) $= \{h_1, h_3, h_5\}$, H (cheap)$= \{h_6, h_9, h_{10}\}$, H (beautiful) $= \{h_2, h_3, h_7\}$, and H (in the green surroundings) $= \{h_5, h_6, h_8\}$.

proposition 1.3.20 [37] Let F_A and G_B be two soft sets over a common universe U. Then
(i) $(F_A \tilde{\vee} G_B)^c = F^c_{\rceil A} \tilde{\wedge} G^c_{\rceil B}$.
(ii) $(F_A \tilde{\wedge} G_B)^c = F^c_{\rceil A} \tilde{\vee} G^c_{\rceil B}$.

Proof. (i) suppose that $F_A \tilde{\vee} G_B = O_{A\times B}$
Therefore $(F_A \tilde{\vee} G_B)^c = (O, A\times B)^c = O^c_{\rceil(A\times B)}$. Now,

$$\begin{aligned} F^c_{\rceil A} \tilde{\wedge} G^c_{\rceil B} &= J_{\rceil A \times \rceil B} \qquad \text{where } J(x,y) = F^c(x) \cap G^c(y) \\ &= J_{\rceil(A\times B)} \end{aligned}$$

Now, take $(\rceil\alpha, \rceil\beta) \in \rceil(A\times B)$. Therefore,

$$\begin{aligned} O^c(\rceil\alpha, \rceil\beta) &= U - O(\alpha,\beta) \\ &= U - [F(\alpha) \cup G(\beta)] \\ &= [U - F(\alpha)] \cap [U - G(\beta)] \\ &= F^c(\rceil\alpha) \cap G^c(\rceil\beta) \\ &= J(\rceil\alpha, \rceil\beta) \end{aligned}$$

Then, O^c and J are same. Hence $(F_A \tilde{\vee} G_B)^c = (F,A)^c \tilde{\wedge} (G,B)^c$.
(ii) suppose that $F_A \tilde{\wedge} G_B = H_{A\times B}$
Therefore $(F_A \tilde{\wedge} G_B)^c = (H, A\times B)^c = H^c_{\rceil(A\times B)}$. Now,

$$\begin{aligned} F^c_{\rceil A} \tilde{\vee} G^c_{\rceil B} &= K_{\rceil A \times \rceil B} \qquad \text{where } K(x,y) = F^c(x) \cup G^c(y) \\ &= K_{\rceil(A\times B)} \end{aligned}$$

Now, take $(\rceil\alpha, \rceil\beta) \in \rceil(A\times B)$. Therefore,

$$\begin{aligned} H^c(\rceil\alpha, \rceil\beta) &= U - H(\alpha, \beta) \\ &= U - [F(\alpha) \cap G(\beta)] \\ &= [U - F(\alpha)] \cup [U - G(\beta)] \\ &= F^c(\rceil\alpha) \cup G^c(\rceil\beta) \\ &= K(\rceil\alpha, \rceil\beta) \end{aligned}$$

Then, H^c and K are same. Hence $(F_A \tilde{\wedge} G_B)^c = (F, A)^c \tilde{\vee} (G, B)^c$. ■

In 2009, M. I. Ali et al. [15] gave some new operations in soft set theory.

Definition 1.3.21 [15] Let F_A and G_B be two soft sets over a common universe U. The restricted intersection of F_A and G_B is defined as the soft set $H_C = F_A \tilde{\sqcap} G_B$, where $C = A \cap B \neq \emptyset$ and $H(x) = F(x) \cap G(x)$ for all $x \in C$.

Definition 1.3.22 [15] The extended intersection of two soft sets F_A and G_B over a common universe U is defined as the soft set $H_C = F_A \tilde{\sqcap} G_B$, where $C = A \cup B$ and for all $x \in C$

$$H(x) = \begin{cases} F(x) & \text{if } x \in A - B \\ G(x) & \text{if } x \in B - A \\ F(x) \cap G(x) & \text{if } x \in A \cap B \end{cases}$$

Definition 1.3.23 [15] Let F_A and G_B be two soft sets over a common universe U. The restricted union of F_A and G_B is defined as the soft set $H_C = F_A \tilde{\cup} G_B$, where $C = A \cap B \neq \emptyset$ and $H(x) = F(x) \cup G(x)$ for all $x \in C$.

F. Feng et al. [26] were first to introduc a generalization of union, AND, OR and intersection of two soft sets, they defined them of a nonempty family of soft sets. After three years, A. Sezgin et al. [45] introduced a generalization of the rest of the operations in soft sets theory.

Definition 1.3.24 [26] The restricted intersection of a nonempty family of soft sets $\{(F_i)_{A_i} : i \in \Lambda\}$ over a common universe U is defined as the soft set $H_B = \tilde{\sqcap}_{i\in\Lambda}(F_i)_{A_i}$, where $B = \cap_{i\in\Lambda}A_i \neq \emptyset$ and $H(x) = \cap_{i\in\Lambda}F_i(x)$, $\Lambda(x) = \{i \in \Lambda : x \in A_i\}$ for all $x \in B$.

Definition 1.3.25 [45] The extended intersection of a nonempty family of soft sets $\{(F_i)_{A_i} : i \in \Lambda\}$ over a common universe U is defined as the soft set $H_B = \tilde{\cap}_{i\in\Lambda}(F_i)_{A_i}$, where $B = \cup_{i\in\Lambda}A_i$ and $H(x) = \cap_{i\in\Lambda}F_i(x)$, $\Lambda(x) = \{i \in \Lambda : x \in A_i\}$ for all $x \in B$.

Definition 1.3.26 [45] The restricted union of a nonempty family of soft sets $\{(F_i)_{A_i} : i \in \Lambda\}$ over a common universe U is defined as the soft set $H_B = \tilde{\cup}_{i\in\Lambda}(F_i)_{A_i}$, where $B = \cap_{i\in\Lambda}A_i \neq \emptyset$ and $H(x) = \cup_{i\in\Lambda}F_i(x)$, $\Lambda(x) = \{i \in \Lambda : x \in A_i\}$ for all $x \in B$.

Definition 1.3.27 [45] The $\wedge$- intersection of a nonempty family of soft sets $\{(F_i)_{A_i} : i \in \Lambda\}$ over a common universe U is defined as the soft set $H_B = \tilde{\wedge}_{i\in\Lambda}(F_i)_{A_i}$, where $B = \Pi_{i\in\Lambda}A_i$ and $H(x) = \cap_{i\in\Lambda}F_i(x)$, $\Lambda(x) = \{i \in \Lambda : x \in A_i\}$ for all $x \in B$.

Definition 1.3.28 [45] The $\vee$- union of a nonempty family of soft sets $\{(F_i)_{A_i} : i \in \Lambda\}$ over a common universe U is defined as the soft set $H_B = \tilde{\vee}_{i\in\Lambda}(F_i)_{A_i}$, where $B = \Pi_{i\in\Lambda}A_i$ and $H(x) = \cup_{i\in\Lambda}F_i(x)$,
$\{i \in \Lambda : x \in A_i\}$ for all $x \in B$.

Definition 1.3.29 [45] The Cartesian product of the nonempty family of soft sets $\{(F_i)_{A_i} : i \in \Lambda\}$ over a common universe U is defined as the soft set $H_B = \tilde{\Pi}_{i\in\Lambda}(F_i)_{A_i}$, where $B = \Pi_{i\in\Lambda}A_i$ and $H(x) = \Pi_{i\in\Lambda}F_i(x)$, $\Lambda(x) = \{i : i \in A_i\}$ for all $x \in B$.

Definition 1.3.30 [26] For a soft set F_A, the set $Supp\ F_A = \{x \in A : F(x) \neq \emptyset\}$ is called the support of the soft set F_A, and the soft set F_A is called a non-null if $Supp\ F_A \neq \emptyset$.

1.4 Fuzzy Structures of Soft Set

In 2001, Maji et al. [36] expended the soft set theory to fuzzy soft set theory. They defined the notion of fuzzy soft set in the following way:
A pair (f, A) is called a fuzzy soft set over U, where f is a mapping given by $f : A \to \tilde{P}(U)$, $\tilde{P}(U) = I^U$, where I^U denotes the collection of all fuzzy subset of U and $I = [0, 1]$. In general, for every $\varepsilon \in A$, $f(\varepsilon) = f_\varepsilon$ is a fuzzy set of U and it is called fuzzy value set of parameter x. The set of all fuzzy soft sets over U with parameters from E is called a fuzzy soft class, and it is denoted by $FS(U, E)$.

Definition 1.4.1 [36] A fuzzy soft set (f, A) over U is called a null fuzzy soft set, denoted by Φ , if $f(\varepsilon)$ is the null fuzzy set $\bar{0}$ of U, where $\bar{0}(x) = 0$ for all $x \in U$. A fuzzy soft set (g, A) over U is called a whole fuzzy soft set, denoted by U, if $g(\varepsilon)$ is the whole fuzzy set $\bar{1}$ of U, where $\bar{1}(x) = 1$ for all $x \in U$.

Definition 1.4.2 [36] Let (f, A) and (g, B) be two fuzzy soft sets over U. We say that (f, A) is a fuzzy soft subset of (g, B) and write
$(f, A) \Subset (g, B)$ if
(i) $A \subseteq B$,
(ii) For any $\varepsilon \in A$, $f(\varepsilon) \subseteq g(\varepsilon)$.

(f, A) and (g, B) are said to be fuzzy soft equal and write
$(f, A) = (g, B)$ if $(f, A) \Subset (g, B)$ and $(g, B) \Subset (f, A)$.

Definition 1.4.3 [36] If (f, A) and (g, B) are two fuzzy soft sets over the same universe U then "(f, A) AND (g, B)" is a fuzzy soft set denoted by $(f, A) \wedge (g, B)$, and is defined by $(f, A) \wedge (g, B) = (h, A \times B)$ where, $h(a, b) = f(a) \cap g(b)$ for all $(a, b) \in A \times B$. Here $\cap$ is the operation of fuzzy intersection.

Definition 1.4.4 [36] If (f, A) and (g, B) are two fuzzy soft sets over the same universe U then "(f, A) OR (g, B)" is a fuzzy soft set denoted by $(f, A)\tilde{\vee}(g, B)$, and is defined by $(f, A)\tilde{\vee}(g, B) = (h, A \times B)$ where, $h(a, b) = f(a) \cup g(b)$ for all $(a, b) \in A \times B$. Here $\cup$ is the operation union of fuzzy set.

To solve decision making problems based on fuzzy soft sets, Feng et al. [27] introduced the following notion called t-level soft sets of fuzzy soft sets.

Definition 1.4.5 Let (f, A) be a fuzzy soft set over U. For each $t \in [0, 1]$, the set $(f, A)^t = (f^t, A)$ is called a t-level soft set of (f, A), where $f^t_\varepsilon = \{x \in U : f_\varepsilon(x) \geq t\}$ for all $\varepsilon \in A$. Clearly, $(f, A)^t$ is a soft set over U.

Definition 1.4.6 [18] Let (f, A) and (g, B) be two fuzzy soft sets over a common universe U with $A \cap B \neq \emptyset$, then their restricted intersection is a fuzzy soft set $(h, A \cap B)$ denoted by $(f, A) \Cap (g, B) = (h, A \cap B)$ where, $h(\varepsilon) = f(\varepsilon) \cap g(\varepsilon)$ for all $\varepsilon \in A \cap B$.

Definition 1.4.7 [18] Let (f, A) and (g, B) be two fuzzy soft sets over U. Then their extended intersection is a fuzzy soft set denoted by (h, C), where $C = A \cup B$ and

$$h(\varepsilon) = \begin{cases} f_\varepsilon & \text{if} \quad \varepsilon \in A - B \\ g_\varepsilon & \text{if} \quad \varepsilon \in B - A \\ f_\varepsilon \cap g_\varepsilon & \text{if} \quad \varepsilon \in A \cap B \end{cases}$$

for all $\varepsilon \in C$. This is denoted by $(h, C) = (f, A)\tilde{\cap}(g, B)$.

Definition 1.4.8 [18] Let (f, A) and (g, B) be two fuzzy soft sets over a common universe U with $A \cap B \neq \emptyset$, then their restricted union is denoted by $(f, A) \Cup (g, B)$ and is defined as $(f, A) \Cup (g, B) = (h, C)$ where $C = A \cap B$ and for all $\varepsilon \in C$, $h(\varepsilon) = f(\varepsilon) \cup g(\varepsilon)$.

Definition 1.4.9 [18] Let (f, A) and (g, B) be two fuzzy soft sets over U. Then their extended union denoted by (h, C), where $C = A \cup B$ and

$$h(\varepsilon) = \begin{cases} f_\varepsilon & \text{if} \quad \varepsilon \in A - B \\ g_\varepsilon & \text{if} \quad \varepsilon \in B - A \\ f_\varepsilon \cup g_\varepsilon & \text{if} \quad \varepsilon \in A \cap B \end{cases}$$

for all $\varepsilon \in C$. This is denoted by $(h, C) = (f, A)\tilde{\cup}(g, B)$.

Definition 1.4.10 The extended product of two fuzzy soft sets (f, A) and (g, B) over U is a fuzzy soft set, denoted by $(f \circ g, C)$, where $C = A \cup B$ and

$$(f \circ g)(\varepsilon) = \begin{cases} f_\varepsilon & \text{if} \quad \varepsilon \in A - B \\ g_\varepsilon & \text{if} \quad \varepsilon \in B - A \\ f_\varepsilon \circ g_\varepsilon & \text{if} \quad \varepsilon \in A \cap B \end{cases}$$

for all $\varepsilon \in C$. This is denoted by $(f \circ g, C) = (f, A)\tilde{\circ}(g, B)$.

Definition 1.4.11 If $A \cap B \neq \emptyset$, then the restricted product (h, C) of two fuzzy soft sets (f, A) and (g, B) over U is defined as the fuzzy soft set, $(h, A \cap B)$ denoted by $(f, A) o_R (g, B)$ where $h(\varepsilon) = f(\varepsilon) \circ g(\varepsilon)$, for all $\varepsilon \in A \cap B$. Here $f(\varepsilon) \circ g(\varepsilon)$ is the product of two fuzzy subsets of U.

Chapter 2

Applications of soft sets in K-algebras

In 1999, Molodtsov introduced the concept of soft set theory as a general mathematical tool for dealing with uncertainty and vagueness. In this chapter, we apply the concept of soft sets to K-algebras and investigate some properties of Abelian soft K-algebras. We also introduce the concept of soft intersection K-subalgebras and investigate some of their properties.

2.1 Soft K-algebras

If K is a K-algebra and A a nonempty set, a set-valued function $F : A \to P(K)$ can be defined by $F(x) = \{y \in K : xRy\}$, $x \in A$, where R is an arbitrary binary relation from A to K, that is, R is a subset of $A \times K$ unless otherwise specified. The pair $F_A = (F, A)$ is then a soft set over K.

Definition 2.1.1 Let F_A be a non-null soft set over K. Then F_A is called a soft K-algebra over K if $F(x)$ is a K-subalgebra of K for all $x \in Supp\ F_A$.

Example 2.1.2 Consider the K-algebra $K = (S_3, ., \odot, e)$ on the sym-

metric group $S_3 = \{e, a, b, x, y, z\}$ where $e = (1), a = (123), b = (132), x = (12), y = (13), z = (23)$, and $\odot$ is given in example 1.1.3.
Let F_A be a soft set over K, where $A = K$ and $F : A \to P(K)$ is set-valued function defined by $F(e) = \{e\}, F(a) = F(b) = \{e, a, b\}, F(x) = \{e, x\}, F(y) = \{e, y\}$, and $F(z) = \{e, z\}$. Then, it is easy to check that $F(e), F(a), F(b), F(x), F(y)$ and $F(z)$ are K-subalgebras of K for all $x \in$ *Supp* F_A. Therefore, F_A is a soft K-algebra over K.

Example 2.1.3 Consider the K-algebra $K = (G, ., \odot, e)$ on the Dihedral group $G = \{e, a, u, v, b, x, y, z\}$ where $u = a^2, v = a^3, x = ab, y = a^2b, z = a^3b$, and $\odot$ is given by the following Cayley's table:

$\odot$	e	a	u	v	b	x	y	z
e	e	v	u	a	b	x	y	z
a	a	e	v	u	x	y	z	b
u	u	a	e	v	y	z	b	x
v	v	u	a	e	z	b	x	y
b	b	x	y	z	e	v	u	a
x	x	y	z	b	a	e	v	u
y	y	z	b	x	u	a	e	v
z	z	b	x	y	v	u	a	e

Let F_A be a soft set over K, where $A = K$ and $F : A \to P(K)$ is set-valued function defined by $F(e) = \{e\}, F(a) = F(v) = \{e, a, u, v\}, F(u) = \{e, u\}, F(b) = \{e, b\}, F(x) = \{e, x\}, F(y) = \{e, y\}$ and $F(z) = \{e, z\}$. Then, it is easy to check that $F(e), F(a), F(v), F(u), F(b), F(x), F(y)$ and $F(z)$ are K-subalgebras of K. Therefore F_A is a soft K-algebra over K.

Lemma 2.1.4 Let F_A be a soft K-algebra over K, then
(i) If $x \in F(x) \Longrightarrow x^{-1} \in F(x)$ for all $x \in A$.
(ii) If $a \odot b \in F(x) \Longrightarrow b \odot a \in F(x)$ for all $a, b \in A$.

Proof. (i) Since F_A is a soft K-algebra over K, then $F(x)$ is a K-subalgebra of K and $e \in F(x)$.
Let $x \in F(x)$. Then $e \odot x \in F(x) \Longrightarrow e.x^{-1} \in F(x) \Longrightarrow x^{-1} \in F(x) \quad \forall x \in A.$

(ii) Since F_A is a soft K-algebra over K, then $F(x)$ is a K-subalgebra of K, let $a \odot b \in F(x)$.

$$\begin{aligned} \text{Then } (a \odot b)^{-1} &\in F(x) \quad \text{by(i)} \\ &\Longrightarrow (a.b^{-1})^{-1} \in F(x) \\ &\Longrightarrow b.a^{-1} \in F(x) \\ &\Longrightarrow b \odot a \in F(x) \quad \forall a, b \in A \end{aligned}$$

■

Proposition 2.1.5 Let $\{(F_i)_{A_i} : i \in \Lambda\}$ be a nonempty family of soft K-algebras over K. Then the restricted intersection $\tilde{\sqcap}_{i\in\Lambda}(F_i)_{A_i}$ is a soft K-algebra over K if it is non-null.

Proof. Let $\{(F_i)_{A_i} : i \in \Lambda\}$ be a nonempty family of soft K-algebras over K. By Definition 1.3.24, we can write $\tilde{\sqcap}_{i\in\Lambda}(F_i)_{A_i} = H_B$, where $B = \cap_{i\in\Lambda} A_i$ and $H(x) = \cap_{i\in\Lambda} F_i(x)$ for all $x \in B$.
Let $x \in Supp\ H_B$. Then $\cap_{i\in\Lambda} F_i(x) \neq \emptyset$, and so we have $F_i(x) \neq \emptyset$ for all $i \in \Lambda$. Since $\{(F_i)_{A_i} : i \in \Lambda\}$ is a nonempty family of soft K-algebras over K, it follows that $F_i(x)$ is a K-subalgebra of K for all $i \in \Lambda$, and its intersection is also a K- subalgebra of K, that is, $H(x) = \cap_{i\in\Lambda} F_i(x)$ is a K-subalgebra of K for all $x \in Supp\ H_B$. Hence $H_B = \tilde{\sqcap}_{i\in\Lambda}(F_i)_{A_i}$ is a soft K-algebra over K. ■

Proposition 2.1.6 Let $\{(F_i)_{A_i} : i \in \Lambda\}$ be a nonempty family of soft K-algebras over K. Then the extended intersection $\tilde{\cap}_{i\in\Lambda}(F_i)_{A_i}$ is a soft K-algebra over K.

Proof. Let $\{(F_i)_{A_i} : i \in \Lambda\}$ be a nonempty family of soft K-algebras over K. By Definition 1.3.25, we can write $\tilde{\cap}_{i\in\Lambda}(F_i)_{A_i} = H_B$, where $B = \cup_{i\in\Lambda} A_i$ and $H(x) = \cap_{i\in\Lambda} F_i(x)$ for all $x \in B$.
Let $x \in Supp\ H_B$. Then $\cap_{i\in\Lambda} F_i(x) \neq \emptyset$ and so we have $F_i(x) \neq \emptyset$ for all $i \in \Lambda$. Since $\{(F_i)_{A_i} : i \in \Lambda\}$ is a nonempty family of soft K-algebras

over K, it follows that $F_i(x)$ is a K-subalgebra of K for all $i \in \Lambda$, and its intersection is also a K- subalgebra of K, that is, $H(x) = \cap_{i\in\Lambda} F_i(x)$ is a K-subalgebra of K for all $x \in Supp\ H_B$. Hence $H_B = \tilde{\cap}_{i\in\Lambda}(F_i)_{A_i}$ is a soft K-algebra over K. ■

Proposition 2.1.7 Let $\{(F_i)_{A_i} : i \in \Lambda\}$ be a nonempty family of soft K-algebras over K. If $F_i(x_i) \subseteq F_j(x_j)$ or $F_j(x_j) \subseteq F_i(x_i)$ for all $i, j \in \Lambda$, $x_i \in A_i$, then the restricted union $\tilde{\cup}_{i\in\Lambda}(F_i)_{A_i}$ is a soft K-algebra over K.

Proof. Suppose that $\{(F_i)_{A_i} : i \in \Lambda\}$ is a nonempty family of soft K-algebras over K. By Definition 1.3.26, we can write $\tilde{\cup}_{i\in\Lambda}(F_i)_{A_i} = H_B$, where $B = \cap_{i\in\Lambda} A_i$ and $H(x) = \cup_{i\in\Lambda} F_i(x)$ for all $x \in B$.
Let $x \in Supp\ H_B$. Since $Supp\ H_B = \cup_{i\in\Lambda} Supp(F_i)_{A_i} \neq \emptyset$, $F_{i_0}(x) \neq \emptyset$ for some $i_0 \in \Lambda$. By assumption $\cup_{i\in\Lambda} F_i(x)$ is a K-subalgebra of K for all $x \in Supp\ H_B$. Hence restricted union $\tilde{\cup}_{i\in\Lambda}(F_i)_{A_i}$ is a soft K-algebra over K. ■

Proposition 2.1.8 Let $\{(F_i)_{A_i} : i \in \Lambda\}$ be a nonempty family of soft K-algebras over K. Then the $\wedge$-intersection $\tilde{\wedge}_{i\in\Lambda}(F_i)_{A_i}$ is a soft K-algebra over K if it is non-null.

Proof. Let $\{(F_i)_{A_i} : i \in \Lambda\}$ be a nonempty family of soft K-algebras over K. By Definition 1.3.27, we can write $\tilde{\wedge}_{i\in\Lambda}(F_i)_{A_i} = H_B$, where $B = \Pi_{i\in\Lambda} A_i$ and $H(x) = \cap_{i\in\Lambda} F_i(x)$ for all $x = (x_i)_{i\in\Lambda} \in B$. Suppose that the soft set H_B is non-null. If $x = (x_i)_{i\in\Lambda} \in Supp\ H_B$, $H(x) = \cap_{i\in\Lambda} F_i(x) \neq \emptyset$. Since $\{(F_i)_{A_i} : i \in \Lambda\}$ is a nonempty family of soft K-algebras over K, nonempty set $F_i(x)$ is a K-subalgebra of K for all $i \in \Lambda$. It follows that $H(x) = \cap_{i\in\Lambda} F_i(x)$ is a K-subalgebra of K for all
$x = (x_i)_{i\in\Lambda} \in Supp\ H_B$. Hence $\wedge$-intersection $\tilde{\wedge}_{i\in\Lambda}(F_i)_{A_i}$ is a soft K-algebra over K. ■

Proposition 2.1.9 Let $\{(F_i)_{A_i} : i \in \Lambda\}$ be a nonempty family of soft

K-algebras over K. If $F_i(x_i) \subseteq F_j(x_j)$ or $F_j(x_j) \subseteq F_i(x_i)$ for all $i, j \in \Lambda$, $x_i \in A_i$, then the $\vee$- union $\tilde{\vee}_{i\in\Lambda}(F_i)_{A_i}$ is a soft K-algebra over K.

Proof. Assume that $\{(F_i)_{A_i} : i \in \Lambda\}$ is a nonempty family of soft K-algebras over K. By Definition 1.3.28, we can write $\tilde{\vee}_{i\in\Lambda}(F_i)_{A_i} = H_B$, where $B = \Pi_{i\in\Lambda}A_i$ and $H(x) = \cup_{i\in\Lambda}F_i(x)$ for all $x = (x_i)_{i\in\Lambda} \in B$.
Let $x = (x_i)_{i\in\Lambda} \in Supp\ H_B$. Then $H(x) = \cup_{i\in\Lambda}F_i(x) \neq \emptyset$, so we have $F_{i_0}(x_{i_0}) \neq \emptyset$ for some $i_0 \in \Lambda$. By assumption $\cup_{i\in\Lambda}F_i(x)$ is a K-subalgebra of K for all $x = (x_i)_{i\in\Lambda} \in Supp\ H_B$. Hence $\vee$- union $\tilde{\vee}_{i\in\Lambda}(F_i)_{A_i}$ is a soft K-algebra over K. ■

Proposition 2.1.10 Let $\{(F_i)_{A_i} : i \in \Lambda\}$ be a nonempty family of soft K-algebras over K. Then the cartesian product $\tilde{\Pi}_{i\in\Lambda}(F_i)_{A_i}$ is a soft K-algebra over $\Pi_{i\in\Lambda}K_i$.

Proof. Let $\{(F_i)_{A_i} : i \in \Lambda\}$ be a nonempty family of soft K-algebras over K. By Definition 1.3.29, we can write $\tilde{\Pi}_{i\in\Lambda}(F_i)_{A_i} = H_B$, where $B = \Pi_{i\in\Lambda}A_i$ and $H(x) = \Pi_{i\in\Lambda}F_i(x)$ for all $x = (x_i)_{i\in\Lambda} \in B$. Suppose that the soft set H_B is non-null. If $x = (x_i)_{i\in\Lambda} \in Supp\ H_B$, then $H(x) = \Pi_{i\in\Lambda}F_i(x) \neq \emptyset$, and so we have $F_i(x_i) \neq \emptyset$ for all $i \in \Lambda$. Since $\{(F_i)_{A_i} : i \in \Lambda\}$ is a nonempty family of soft K-algebras over K, nonempty set $F_i(x)$ is a K-subalgebra of K for all $i \in \Lambda$. It follows that $H(x) = \Pi_{i\in\Lambda}F_i(x)$ is a K-subalgebra of K for all $x = (x_i)_{i\in\Lambda} \in Supp\ H_B$. Hence the cartesian product $\tilde{\Pi}_{i\in\Lambda}(F_i)_{A_i}$ is a soft K-algebra over $\Pi_{i\in\Lambda}K_i$. ■

Definition 2.1.11 Let F_A be a soft K-algebra over K.
(i) F_A is called the trivial soft K-algebra over K if $F(x) = \{e\}$ for all $x \in A$.
(ii) F_A is called the whole soft K-algebra over K if $F(x) = K$ for all $x \in A$.

Definition 2.1.12 Let F_A be a soft set over K. The inverse of F_A is denoted by F_A^{-1} and is defined as follows $F_A^{-1} = \{(F(a))^{-1} : a \in A\}$, where $(F(a))^{-1}$ is called the inverse of $F(a)$ and is defined as $(F(a))^{-1} = \{x^{-1} : x \in F(a)\}$.

Definition 2.1.13 The restricted product H_C of two soft K-algebras F_A and G_B over K is denoted by the soft set $H_C = F_A \hat{\circ} G_B$ where $C = A \cap B$ and H is a set valued function from C to $P(K)$ and is defined as $H(c) = F(c)G(c)$ for all $c \in C$. The soft set H_C is called the restricted soft product of F_A and G_B over K.

Theorem 2.1.14 Let F_A and G_B be any two soft sets over K. Then $(F_A \hat{\circ} G_B)^{-1} = G_B^{-1} \hat{\circ} F_A^{-1}$.

Proof. Suppose that the inverse of restricted soft product of F_A and G_B denoted by $(F_A \hat{\circ} G_B)^{-1} = H_C$ is defined as $H(c) = (F(c)G(c))^{-1}$ for all $c \in C$ and $G_B^{-1} \hat{\circ} F_A^{-1} = L_C$ and is defined as $L(c) = (G(c))^{-1}(F(c))^{-1}$ for all $c \in C$. But then $(F(c)G(c))^{-1} = (G(c))^{-1}(F(c))^{-1}$ for all $c \in C$. This implies that $L(c) = H(c)$ for all $c \in C$.
Thus $(F_A \hat{\circ} G_B)^{-1} = G_B^{-1} \hat{\circ} F_A^{-1}$. ■

Theorem 2.1.15 If F_A is a soft K-algebra over K, then $F_A^{-1} = F_A$.

The converse of above theorem is not true in general, and it can be seen in the following example.

Example 2.1.16 Consider the K-algebra $K = (G, ., \odot, e)$ on the klien four group $G = \{e, a, b, c\}$ and $\odot$ is given by the following Cayley's table:

$\odot$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Let F_A be a soft set over K, where $A = K$ and $F : A \to P(K)$ is set-valued function defined by $F(e) = \{a, b, c\}, F(a) = \{b, c\}, F(b) = \{a, c\}$ and $F(c) = \{a, b\}$. Then $(F(e))^{-1} = \{a, b, c\}, (F(a))^{-1} = \{b, c\}, (F(b))^{-1} = \{a, c\}$

and $(F(c))^{-1} = \{a, b\}$. Therefore we find that $F(\alpha) = (F(\alpha))^{-1}$ for all $\alpha \in A$. Hence $F_A^{-1} = F_A$, but F_A is not soft K-algebra over K since each $F(\alpha)$ is not a K-subalgebra of K.

Definition 2.1.17 A soft K-algebra F_A over K is said to be abelian soft K-algebra over K if each $F(\alpha)$ is an abelian K-subalgebra of K for all $\alpha \in A$.

Example 2.1.18 Let F_A be a soft K-algebra over K which is given in Example 2.1.2 Then it is easy to verify that each $F(x)$ is an abelian K-subalgebra of K for all $x \in A$. Hence F_A is an abelian soft K-algebra over K.

Definition 2.1.19 Let F_A be a soft K-algebra over K and H_B be a soft K-subalgebra of F_A. Then we say that H_B is an abelian soft K-subalgebra of F_A if $H(x)$ is an abelian K-subalgebra of $F(x)$ for all $x \in B$.

Example 2.1.20 Let F_A be a soft K-algebra over K which is given in Example 2.1.2 , and let H_B be a soft set over K, where $B = A_3$ and $H : B \to P(K)$ the set-valued function defined by $H(e) = \{e\}$, $H(a) = \{e, a, b\}$ and $H(b) = \{e, a, b\}$ are abelian K-subalgebras of $F(e), F(a)$ and $F(b)$, respectively .Hence H_B is an abelian soft K-subalgebra of F_A.

Theorem 2.1.21 Let F_A be an abelian soft K-algebra over K and G_B be a soft K-algebra over K. Then their restricted intersection $F_A \tilde{\sqcap} G_B$ is an abelian soft K-algebra over K for all $c \in A \cap B$.

Proof. By Definition 1.3.24, we can write $F_A \tilde{\sqcap} G_B = H_C$, where $C = A \cap B$ and $H(x) = F(x) \cap G(x)$. Then $H(x)$ is an abelian K-subalgebra of K for all $x \in C$. Hence $H_C = F_A \tilde{\sqcap} G_B$ is an abelian soft K-algebra over K ■

2.2 Homomorphism of Soft K-algebras

Definition 2.2.1 Let K_1, K_2 be two K-algebras and $\varphi : K_1 \to K_2$ a mapping of K-algebras. If F_A and G_B are soft sets over K_1 and K_2 respectively, then $\varphi(F_A)$ is a soft set over K_2 where $\varphi(F) : E \to P(K_2)$ is defined by $\varphi(F)(x) = \varphi(F(x))$ for all $x \in E$ and $\varphi^{-1}(G_B)$ is a soft set over K_1 where $\varphi^{-1}(G) : E \to P(K_1)$ is defined by $\varphi^{-1}(G)(y) = \varphi^{-1}(G(y))$for all $y \in E$.

Definition 2.2.2 Let F_A and H_B be two soft sets over K-algebras K_1 and K_2, respectively, and let $\varphi : K_1 \to K_2$ and $\phi : A \to B$ be two functions. Then we say that (φ, ϕ) is a soft homomorphism, if the following conditions are satisfied:
(i) φ is a homomorphism from K_1 onto K_2,
(ii) ϕ is onto,
(iii) $\varphi(F(x)) = H(\phi(x))$.

In this definition, if φ is an isomorphism from K_1 to K_2 and ϕ is a one-to-one mapping from A on to B, then we say that (φ, ϕ) is a soft isomorphism and that F_A is soft isomorphic to H_B. Notation, $F_A \simeq H_B$

Example 2.2.3 Let $G = \{e, a, b, c\}$ be a klein four group. Consider a K-algebra on G and $\odot$ is given in example 2.1.16. Let F_A be a soft set over K, where $A = K$ and $F : A \to P(K)$ the set-valued function defined by $F(x) = \{r \in K : xRr \iff x \odot r \in A_x\}$ where $A_x = \{e, x, x^{-1}\}$. Then F_A is a soft K-algebra over K. Let G_A be a soft set over K, where $A = K$ and $G : A \to P(K)$ the set-valued function defined by $G(x) = \{r \in K : xRr \iff x^n = r, n \in N\}$.Then G_A is a soft K-algebra over K. Let $\varphi : K \to K$ be the mapping defined by $\varphi(x) = x$. It is clear that φ is a K-homomorphism. Consider the mapping $\phi : A \to A$ given by $\phi(x) = x^3$. Then one can easily verify that $\varphi(F(x)) = G(\phi(x))$

for all $x \in A$. Hence (φ, ϕ) is a soft homomorphism from K to K.

Example 2.2.4 Consider the K-algebra $K = (A_3, ., \odot .e)$ with the following Cayley's table:

$\odot$	e	a	b
e	e	b	a
a	a	e	b
b	b	a	e

Let F_A be a soft set over K, where $A = K$ and $F : A \to P(K)$ the set-valued function defined by $F(x) = \{r \in K : xRr \iff x \odot r \in A_x\}$ where $A_x = \{e, x, x^{-1}\}$. Then F_A is a soft K-algebra over K. Let G_A be a soft set over K, where $A = K$ and $G : A \to P(K)$ the set-valued function defined by $G(x) = \{r \in K : xRr \iff x^n = r, n \in N\}$. Then G_A is a soft K-algebra over K. Let $\varphi : K \to K$ be the mapping defined by $\varphi(x) = x^4$. It is clear that φ is a K-homomorphism. Consider the mapping $\phi : A \to A$ given by $\phi(x) = x$. Then one can easily verify that $\varphi(F(x)) = G(\phi(x))$ for all $x \in A$. Hence (φ, ϕ) is a soft homomorphism from K to K.

Proposition 2.2.5 Let $\varphi : K_1 \to K_2$ be an onto homomorphism of K-algebras and F_A, G_B two soft K-algebras over K_1 and K_2 respectively.
(i) The soft function (φ, I_A) from F_A to H_A is a soft homomorphism from K_1 to K_2 , where $I_A : A \to A$ is the identity mapping and the set-valued function $H : A \to P(K_2)$ is defined by $H(x) = \varphi(F(x))$ for all $x \in A$.
(ii) If $\varphi : K_1 \to K_2$ is an isomorphism, then the soft function (φ^{-1}, I_B) from G_B to S_B is a soft homomorphism from K_2 to K_1, where $I_B : B \to B$ is the identity mapping and the set-valued function $S : B \to P(K_1)$ is defined by $S(x) = \varphi^{-1}(G(x))$ for all $x \in B$

Proof. The proofs follow from the Definitions 2.2.2. ■

Proposition 2.2.6 Let K_1, K_2 and K_3 be K-algebras and F_A, G_B and

H_C soft K-algebras over K_1, K_2, and K_3 respectively. Let the soft function (φ, ϕ) from F_A to G_B be a soft homomorphism from K_1 to K_2, and the soft function (φ', ϕ') from G_B to H_C a soft homomorphism from K_2 to K_3 .Then the soft function $(\varphi' \circ \varphi, \phi' \circ \phi)$ from F_A to H_C is a soft homomorphism from K_1 to K_3.

Proof. The proof follow from the Definitions 2.2.2 and properties of the homomorphism. ■

Theorem 2.2.7 Let K_1 and K_2 be K-algebras and F_A, G_B soft sets over K_1 and K_2 respectively. If F_A is a soft K-algebra over K_1 and $F_A \simeq G_B$, then G_B is a soft K-algebra over K_2 .

Proof. The proof follow from the Definitions 2.2.2 and properties of the ismorphism. ■

Definition 2.2.8 Let F_A and G_B be two soft K-algebras over K_1 and K_2, respectively. Then the Cartesian product of soft K-algebras F_A and G_B is denoted by $F_A \tilde{\times} G_B = (U, A \times B)$ and U is defined as

$$U(a,b) = F(a) \times G(b) \;\; for \;\; all \;\; (a,b) \in A \times B.$$

Theorem 2.2.9 Let F_A and H_B be two soft K-algebras over K_1 and K_2, respectively. Then:
(1) the Cartesian product $F_A \tilde{\times}\ H_B$ is a soft K-algebra over $K_1 \times K_2$,
(2) $F_A \tilde{\times}\ H_B$ is soft isomorphic to $H_B \tilde{\times}\ F_A$.

Proof. First part is straightforward. We will prove second part. Now we show that $(\varphi, \phi) : F_A \tilde{\times} H_B \to H_B \tilde{\times}\ F_A$ is a soft isomorphism, that is, $(\varphi, \phi) : (U, A \times B) \to (W, B \times A)$ is a soft isomorphism where $W(b,a)$ is defined as $W(b,a) = H(b) \times F(a)$. We prove three conditions.
(i) We show that $\varphi : K_1 \times K_2 \to K_2 \times K_1$ is an isomorphism. Let φ be a function defined by $\varphi(r,s) = (s,r)$. Then obviously φ is an

isomorphism.
(ii) We now show that $\phi : A \times B \to B \times A$ is a bijective mapping. The mapping ϕ is defined by $\phi(a,b) = (b,a)$ then obviously ϕ is a bijective mapping.
(iii)

$$\begin{aligned}
\varphi(U(a,b)) &= \varphi(F(a) \times H(b)) \\
&= \varphi(\{(r,s) : r \in F(a), s \in H(b)\}) \\
&= \{(s,r) : s \in H(b), r \in F(a)\} \\
&= H(b) \times F(a) \\
&= W(b,a) \\
&= W(\phi(a,b))
\end{aligned}$$

for all $(a,b) \in A \times B$. This implies that $(\varphi, \phi) : F_A \tilde{\times} H_B \to H_B \tilde{\times}\ F_A$ is a soft isomorphism. Hence, $F_A \tilde{\times} H_B \simeq H_B \tilde{\times}\ F_A$. ■

2.3 Soft Intersection K-subalgebras

Definition 2.3.1 Let $K = E$ be a K-algebra and let A be a subset of K. Let F_A be a soft set over K. Then, F_A is called a soft intersection K-subalgebra over K if it satisfies the following condition:

$$F(x \odot y) \supseteq F(x) \cap F(y)$$

for all $x, y \in A$.

Example 2.3.2 Assume that $K = \mathbb{Z}$ is the universal set. Let $A = G = \{e, a, a^2\}$ be the cyclic group of order 3. Then $(G, ., \odot, e)$ is a K-algebra K, and $\odot$ is given by the following Cayley's table:

$\odot$	e	a	a^2
e	e	a^2	a
a	a	e	a^2
a^2	a^2	a	e

Let F_A be a soft set over K and the set-valued function defined by $F(e) = \mathbb{Z}$ and $F(a) = F(a^2) = \{-2, -1, 0, 1, 2\}$. It is easy to check that F_A is a soft intersection K-subalgebra over K.

Proposition 2.3.3 Let K be a K-algebra and let A and B be K-subalgebras of K. If F_A and G_B are soft intersection K-subalgebras over K. Then $F_A \tilde{\wedge} G_B$ is a soft intersection K-subalgebra over K, where $F_A \tilde{\wedge} G_B$ is defined by $F_A \tilde{\wedge} G_B(x, y) = F(x) \cap G(y)$ for all $(x, y) \in A \times B$.

Proof. Let $(x_1, y_1), (x_2, y_2) \in A \times B$. Then

$$\begin{aligned}
(F_A \tilde{\wedge} G_B)((x_1, y_1) \odot (x_2, y_2)) &= (F_A \tilde{\wedge} G_B)((x_1 \odot x_2, y_1 \odot y_2)) \\
&= F(x_1 \odot x_2) \cap G(y_1 \odot y_2) \\
&\supseteq (F(x_1) \cap F(x_2)) \cap (G(y_1) \cap G(y_2)) \\
&= (F(x_1) \cap G(y_1)) \cap (F(x_2) \cap G(y_2)) \\
&= (F_A \tilde{\wedge} G_B)(x_1, y_1) \cap (F_A \tilde{\wedge} G_B)(x_2, y_2).
\end{aligned}$$

Hence $F_A \tilde{\wedge} G_B$ is a soft intersection K-subalgebra over K. ■

Theorem 2.3.4 Let $\{(F_i)_{A_i} : i \in \Lambda\}$ be a family of soft intersection K-subalgebras over K. Then $\tilde{\wedge}_{i \in \Lambda} (F_i)_{A_i}$ is a soft intersection K-subalgebra over K.

Proof. The proof follow from the Proposition 2.3.3. ■

Proposition 2.3.5 Let K be a K-algebra and let A be a K-subalgebra of K. If F_A and G_A are soft intersection K-subalgebras over K. Then $F_A \tilde{\cap} G_A$ is a soft intersection K-subalgebra over K, where $F_A \tilde{\cap} G_A$ is defined by $F_A \tilde{\cap} G_A(x) = F(x) \cap G(x)$ for all $x \in A$.

Proof. Let $x, y \in A$. Then

$$\begin{aligned}(F_A \tilde{\cap} G_A)(x \odot y) &= F(x \odot y) \cap G(x \odot y) \\ &\supseteq (F(x) \cap F(y)) \cap (G(x) \cap G(y)) \\ &= (F(x) \cap G(x)) \cap (F(y) \cap G(y)) \\ &= (F_A \tilde{\cap} G_A)(x) \cap (F_A \tilde{\cap} G_A)(y).\end{aligned}$$

Hence $F_A \tilde{\cap} G_A$ is a soft intersection K-subalgebras over K. ■

Theorem 2.3.6 Let $\{(F_i)_{A_i} : i \in \Lambda\}$ be a family of soft intersection K-subalgebras over K. Then $\tilde{\cap}_{i \in \Lambda}(F_i)_{A_i}$ is a soft intersection K-subalgebra over K.

Proof. The proof follow from the Proposition 2.3.5. ■

Proposition 2.3.7 Let K be a K-algebra and let A and B be K-subalgebras of K. If F_A and G_B are soft intersection K-subalgebras over K. Then $F_A \tilde{\times} G_B$ is a soft intersection K-subalgebra over K, where $F_A \tilde{\times} G_B$ is defined by $F_A \tilde{\times} G_B(x, y) = F(x) \times G(y)$ for all $(x, y) \in A \times B$.

Proof. Let $(x_1, y_1), (x_2, y_2) \in A \times B$. Then

$$\begin{aligned}(F_A \tilde{\times} G_B)((x_1, y_1) \odot (x_2, y_2)) &= (F_A \tilde{\times} G_B)((x_1 \odot x_2, y_1 \odot y_2)) \\ &= F(x_1 \odot x_2) \times G(y_1 \odot y_2) \\ &\supseteq (F(x_1) \cap F(x_2)) \times (G(y_1) \cap G(y_2)) \\ &= (F(x_1) \times G(y_1)) \cap (F(x_2) \times G(y_2)) \\ &= (F_A \tilde{\times} G_B)(x_1, y_1) \cap (F_A \tilde{\times} G_B)(x_2, y_2).\end{aligned}$$

Hence $F_A \tilde{\times} G_A$ is a soft intersection K-subalgebras over K. ■

Theorem 2.3.8 Let $\{(F_i)_{A_i} : i \in \Lambda\}$ be a family of soft intersection K-subalgebras over K. Then $\tilde{\Pi}_{i \in \Lambda}(F_i)_{A_i}$ is a soft intersection K-subalgebra over K.

Proof. The proof follow from the Proposition 2.3.7. ■

Definition 2.3.9 Let F_A and G_B be soft K-algebras over K. Then F_A is a soft K-subalgebra of G_B if
(i) $A \subset B$ and
(ii) $F(x)$ is a K-subalgebra of $G(x)$ for all $x \in A$. We write $F_A \tilde{\leq} G_B$.

Proposition 2.3.10 Let K be a K-algebra and let A, B and C be K-subalgebras of K. If F_A, G_B and F_C are soft intersection K-subalgebras over K, $F_A \tilde{\leq} G_B$ and $F_C \tilde{\leq} G_B$, then $F_A \tilde{\cap} F_C \tilde{\leq} G_B$ over K.

Proof. Since $F_A \tilde{\leq} G_B$, then by Definition 2.3.9, $A \subset B$ and $F(x)$ is a K-subalgebra of $G(x)$ for all $x \in A$. Also, since $F_C \tilde{\leq} G_B$, then $C \subset B$ and $F(x)$ is a K-subalgebra of $G(x)$ for all $x \in C$.By Definition 1.3.22, we can write $F_A \tilde{\cap} F_C = F_{(A \cup B)}$ where

$$H(x) = \begin{cases} F(x) & \text{if } x \in A - C \subset B \\ F(x) & \text{if } x \in C - A \subset B \\ F(x) \cap F(x) = F(x) & \text{if } x \in A \cap C \subset B \end{cases}$$

Hence we can see that $A \cup C \subset B$ and $F(x)$ is a K-subalgebra of $G(x)$ for all $x \in A \cup B$. Hence $F_A \tilde{\cap} F_C \tilde{\leq} G_B$ over K. ■

Definition 2.3.11 [19] Let F_A and G_B be two soft sets over the common universe U and let φ be a function from A to B. Then, soft image of F_A under φ denoted by $\varphi(F_A)$ is a soft set over U defined by

$$\varphi(F)(y) = \begin{cases} \cup\{F(x) : x \in A \text{ and } \varphi(x) = y\} & \text{if } \varphi^{-1}(y) \neq \emptyset, \\ \emptyset & \text{otherwise} \end{cases}$$

for all $y \in B$, and soft pre-image (or soft inverse image) of G_B under φ denoted by $\varphi^{-1}(G_B)$ is a soft set over U defined by $\varphi^{-1}(G)(x) = G(\varphi(x))$ for all $x \in A$.

Theorem 2.3.12 Let K be a K-algebra and A, B are K-subalgebras of K. Let φ be a K-homomorphism from A to B. If G_B is a soft intersection K-subalgebra over K. Then $\varphi^{-1}(G_B)$ is a soft intersection K-subalgebra over K.

Proof. Let $x, y \in A$. Then

$$\begin{aligned}\varphi^{-1}(G)(x \odot y) &= G(\varphi(x \odot y)) = G(\varphi(x) \odot \varphi(y)) \\ &\supseteq (G(\varphi(x)) \cap G(\varphi(y)) \\ &= \varphi^{-1}(G)(x) \cap \varphi^{-1}(G)(y).\end{aligned}$$

Hence $\varphi^{-1}(G_B)$ is a soft intersection K-subalgebra over K. ∎

Theorem 2.3.13 Let K be a K-algebra and A, B are K-subalgebras of K and let φ be a K-ismorphism from A to B. If F_A is a soft intersection K-subalgebra over K. Then $\varphi(F_A)$ is a soft intersection K-subalgebra over K.

Proof. Since φ is surjective, there exist $x, y \in A$ such that $a = \varphi(x)$ and $b = \varphi(y)$ for all $a, b \in B$. Then

$$\begin{aligned}\varphi(F)(x \odot y) &= \cup\{F(z) : z \in A, \varphi(z) = a \odot b\} \\ &= \cup\{F(x \odot y) : x, y \in A, \varphi(x) = a, \varphi(y) = b\} \\ &\supseteq \cup\{F(x) \cap F(y) : x, y \in A, \varphi(x) = a.\varphi(y) = b\} \\ &= (\cup\{F(x) : x \in A, \varphi(x) = a\}) \cap (\cup\{F(y) : y \in A, \varphi(y) = b\}) \\ &= \varphi(F)(x) \cap \varphi(F)(y).\end{aligned}$$

Hence $\varphi(F_A)$ is a soft intersection K-subalgebra over K. ∎

Chapter 3

Fuzzy Soft K-algebras

Fuzzy sets and soft sets are two different methods for representing uncertainty. In this chapter we apply these methods in combination to study uncertainty in K-algebras. We introduce the concept of fuzzy soft K-subalgebras and investigate some of their properties. We discuss fuzzy soft images and fuzzy soft inverse images of fuzzy soft K-subalgebras. We introduce the notion of an $(\in, \in \vee q)$-fuzzy soft K-subalgebra which is a generalization of a fuzzy soft K-subalgebra. We also introduce $(\in_\alpha, \in_\alpha \vee q_\beta)$-fuzzy soft K-subalgebras and describe some of their properties.

3.1 Fuzzy soft K-algebras

Definition 3.1.1 Let (f, A) be a fuzzy soft set over K. Then (f, A) is said to be a fuzzy soft K-subalgebra over K if $f(\varepsilon)$ is a fuzzy K-subalgebra of K for all $\varepsilon \in A$, that is, a fuzzy soft set (f, A) over K is called a fuzzy soft K-subalgebra of K if

$$f_\varepsilon(x \odot y) \geq min\{f_\varepsilon(x), f_\varepsilon(y)\} \text{ for all } x, y \in G.$$

Definition 3.1.2 Let (f, A) and (g, B) be fuzzy soft K-subalgebras

over K. Then (f, A) is a fuzzy soft K-subalgebra of (g, B) if
(i) $A \subset B$ and
(ii) $f(\varepsilon)$ is a fuzzy K-subalgebra of $g(\varepsilon)$ for all $\varepsilon \in A$.

Example 3.1.3 Consider the K-algebra $K = (G, ., \odot, e)$ where $G = \{e, a, a^2, a^3\}$ is the cyclic group of order 4 and $\odot$ is given by the following Cayley's table:

$\odot$	e	a	a^2	a^3
e	e	a^3	a^2	a
a	a	e	a^3	a^2
a^2	a^2	a	e	a^3
a^3	a^3	a^2	a	e

Let $A = \{e_1, e_2, e_3\}$ and $f : A \to \tilde{P}(K)$ be a set-valued function defined by

$$f(e_1) = \{(e, 0.7), (a, 0.3), (a^2, 0.6), (a^3, 0.3)\}$$

$$f(e_2) = \{(e, 0.6), (a, 0.2), (a^2, 0.5), (a^3, 0.2)\}$$

$$f(e_3) = \{(e, 0.7), (a, 0.1), (a^2, 0.3), (a^3, 0.1)\}.$$

Let $B = \{e_2, e_3\}$ and $g : B \to \tilde{P}(K)$ be a set-valued function defined by

$$g(e_2) = \{(e, 0.5), (a, 0.2), (a^2, 0.4), (a^3, 0.2)\},$$

$$g(e_3) = \{(e, 0.6), (a, 0.1), (a^2, 0.3), (a^3, 0.1)\}.$$

(1) (f, A) and (g, B) are fuzzy soft sets over K and by routine calculations, it is easy to check that $f(\varepsilon)$ and $g(\varepsilon)$ are fuzzy K-subalgebras for $\varepsilon \in A$ and $\varepsilon \in B$, respectively. Hence (f, A) and (g, B) are fuzzy soft K-subalgebras over K.
(2) Clearly, $B \subset A$ and $g(\varepsilon)$ is fuzzy K-subalgebra of $f(\varepsilon)$ for all $\varepsilon \in B$. Hence (g, B) is a fuzzy soft K-subalgebra of (f, A).

Proposition 3.1.4 Let (f, A) and (g, B) be fuzzy soft K-subalgebras over K, then $(f, A)\tilde{\cap}(g, B)$ is a fuzzy soft K-subalgebra over K.

Proof. Using definition 1.4.7, we can write $(f,A)\tilde{\cap}(g,B)=(h,C)$ where $C=A\cup B$ and

$$h(\varepsilon)=\begin{cases} f_\varepsilon & \text{if} \quad \varepsilon\in A-B \\ g_\varepsilon & \text{if} \quad \varepsilon\in B-A \\ f_\varepsilon\cap g_\varepsilon & \text{if} \quad \varepsilon\in A\cap B \end{cases}$$

for all $\varepsilon\in C$. Note that $h:C\to\tilde{P}(K)$ is a mapping, and therefore (h,C) is a fuzzy soft set over K. Since (f,A) and (g,B) are fuzzy soft K-subalgebras over K, it follows that $h_\varepsilon=f_\varepsilon$ if $\epsilon\in A-B$ or $h_\varepsilon=g_\varepsilon$ if $\epsilon\in B-A$ or $h_\varepsilon=f_\varepsilon\cap g_\varepsilon$ if $\epsilon\in A\cap B$ and in all cases h_ε is a fuzzy K-subalgebra of K for all $\varepsilon\in C$. Hence $(h,C)=(f,A)\tilde{\cap}(g,B)$ is a fuzzy soft K-subalgebra over K. ■

Proposition 3.1.5 Let (f,A) and (g,B) be fuzzy soft K-subalgebras over K, then $(f,A)\wedge(g,B)$ is a fuzzy soft K-subalgebra over K.

Proof. Using definition 1.4.3, we can write $(f,A)\wedge(g,B)=(h,A\times B)$ where $h(a,b)=f(a)\cap g(b)$ for all $(a,b)\in A\times B$. Since $f(a)$ and $g(b)$ are fuzzy K-subalgebras of K, the intersection $f(a)\cap g(b)$ is also a fuzzy K-subalgebras of K. Hence $h(a,b)$ is a fuzzy K-subalgebras of K for all $(a,b)\in A\times B$ and therefore $(h,A\times B)=(f,A)\wedge(g,B)$ is a fuzzy soft K-subalgebra over K. ■

Proposition 3.1.6 Let (f,A) and (g,B) be fuzzy soft K-subalgebras over K. If $A\cap B=\emptyset$ then $(f,A)\tilde{\cup}(g,B)$ is a fuzzy soft K-subalgebra over K.

Proof. Using definition 1.4.9, we can write $(f,A)\tilde{\cup}(g,B)=(h,C)$ where $C=A\cup B$ and

$$h(\varepsilon)=\begin{cases} f_\varepsilon & \text{if} \quad \varepsilon\in A-B \\ g_\varepsilon & \text{if} \quad \varepsilon\in B-A \\ f_\varepsilon\cup g_\varepsilon & \text{if} \quad \varepsilon\in A\cap B \end{cases}$$

for all $\varepsilon\in C$. Since $A\cap B=\emptyset$, either $\varepsilon\in A-B$ or $\varepsilon\in B-A$ for all $\varepsilon\in C$.

If $\varepsilon \in A - B$ then $h_\varepsilon = f_\varepsilon$ is a fuzzy K-subalgebra of K since (f, A) is a fuzzy soft K-subalgebra over K. If $\varepsilon \in B - A$ then $h_\varepsilon = g_\varepsilon$ is a fuzzy K-subalgebra of K since (g, B) is a fuzzy soft K-subalgebra over K. Hence $(h, C) = (f, A)\tilde{\cup}(g, B)$ is a fuzzy soft K-subalgebra over K. ■

Proposition 3.1.7 Let (f, A) and (g, B) be fuzzy soft K-subalgebras over K. If $f(x) \subseteq g(x)$ for all $x \in A$, then (f, A) is a fuzzy soft K-subalgebra of (g, B).

Proof. The proof follow from the Definitions 3.1.2. ■

Theorem 3.1.8 Let (f, A) be fuzzy soft K-subalgebra over K and let $\{(h_i, B_i)|i \in I\}$ be a nonempty family of fuzzy soft K-subalgebras of (f, A). Then
(a) $\tilde{\cap}_{i\in I}(h_i, B_i)$ is a fuzzy soft K-subalgebra of (f, A),
(b) $\wedge_{i\in I}(h_i, B_i)$ is a fuzzy soft K-subalgebra of (f, A),
(c) If $B_i \cap B_j = \emptyset$ for all $i, j \in I$, then $\tilde{\cup}_{i\in I}(h_i, B_i)$ is a fuzzy soft K-subalgebra of (f, A).

Proof. The proofs follow from the Definitions 1.4.7, 1.4.3, 1.4.9 and Propositions 3.1.4, 3.1.5 and 3.1.6. ■

Theorem 3.1.9 Let (f, A) be a fuzzy soft set over K. (f, A) is a fuzzy soft K-subalgebra if and only if $(f, A)^t$ is a soft K-algebra over K for each $t \in [0, 1]$.

Proof. Suppose that (f, A) is a fuzzy soft K-subalgebra. For each $t \in [0, 1]$, $\varepsilon \in A$ and $x_1, x_2 \in f_\varepsilon^t$ then $f_\varepsilon(x_1) \geq t$ and $f_\varepsilon(x_2) \geq t$. From Definition 3.1.1, it follows that f_ε is a fuzzy K-subalgebra of K. Thus $f_\varepsilon(x_1 \odot x_2) \geq min\{f_\varepsilon(x_1), f_\varepsilon(x_2)\}$, $f_\varepsilon(x_1 \odot x_2) \geq t$. This implies that $x_1 \odot x_2 \in f_\varepsilon^t$, i.e., f_ε^t is a K-subalgebra of K. According to Definition 1.4.5, $(f, A)^t$ is a soft K-algebra over K for each $t \in [0, 1]$. Conversely, assume that $(f, A)^t$ is a soft K-algebra over K for each $t \in [0, 1]$. For each $\varepsilon \in A$ and $x_1, x_2 \in K$, let $t = min\{f_\varepsilon(x_1), f_\varepsilon(x_2)\}$, then $x_1, x_2 \in f_\varepsilon^t$. Since f_ε^t is a K-subalgebra of K, then $x_1 \odot x_2 \in f_\varepsilon^t$. This means that $f_\varepsilon(x_1 \odot x_2) \geq min\{f_\varepsilon(x_1), f_\varepsilon(x_2)\}$, i.e., f_ε is a fuzzy K-subalgebra of K. According to Definition 3.1.1, (f, A) is a fuzzy soft K-subalgebra over K. ■

Definition 3.1.10 [17] Let $\phi : X \to Y$ and $\psi : A \to B$ be two functions, A and B are parametric sets from X and Y , respectively. Then the pair (ϕ, ψ) is called a fuzzy soft function from X to Y.

Definition 3.1.11 Let (f, A) and (g, B) be two fuzzy soft sets over K_1 and K_2, respectively and let (ϕ, ψ) be a fuzzy soft function from K_1 to K_2.
(i) The image of (f, A) under the fuzzy soft function (ϕ, ψ), denoted by $(\phi, \psi)(f, A)$, is the fuzzy soft set on K_2 defined by
$(\phi, \psi)(f, A) = (\phi\ (f), \psi(A))$, where for all $k \in \psi(A), y \in K_2$

$$\phi(f)_k(y) = \begin{cases} \vee_{\phi(x)=y} \vee_{\psi(a)=k} f_a(x) & \text{if } x \in \psi^{-1}(y), \\ 0 & \text{otherwise} \end{cases}$$

(ii) The preimage of (g, B) under the fuzzy soft function (ϕ, ψ), denoted by $(\phi, \psi)^{-1}(g, B)$, is the fuzzy soft set over K_1 defined by
$(\phi, \psi)^{-1}(g, B) = (\phi^{-1}(g), \psi^{-1}(B))$ where $\phi^{-1}(g)_a(x) = g_{\psi(a)}(\phi(x))$,for all $a \in \psi^{-1}(A)$, for all $x \in K_1$.

Definition 3.1.12 Let (ϕ, ψ) be a fuzzy soft function from K_1 to K_2. If ϕ is a homomorphism from K_1 to K_2 then (ϕ, ψ) is said to be fuzzy soft homomorphism, If ϕ is an isomorphism from K_1 to K_2 and ψ is one-to-one mapping from A onto B then (ϕ, ψ) is said to be fuzzy soft isomorphism.

Theorem 3.1.13 Let (g, B) be a fuzzy soft K-subalgebra over K_2 and let (ϕ, ψ) be a fuzzy soft homomorphism from K_1 to K_2. Then $(\phi, \psi)^{-1}(g, B)$ is a fuzzy soft K-subalgebra over K_1.

Proof. Let $x_1, x_2 \in K_1$, then

$$\begin{aligned}
\phi^{-1}(g_\varepsilon)(x_1 \odot x_2) &= g_{\psi(\varepsilon)}(\phi(x_1 \odot x_2)) \\
&= g_{\psi(\varepsilon)}(\phi(x_1) \odot \phi(x_2)) \\
&\geq \min\{g_{\psi(\varepsilon)}(\phi(x_1), g_{\psi(\varepsilon)}\phi(x_2)\} \\
&= \min\{\phi^{-1}(g_\varepsilon)(x_1), \phi^{-1}(g_\varepsilon)(x_2)\}
\end{aligned}$$

Hence $(\phi, \psi)^{-1}(g, B)$ is a fuzzy soft K-subalgebra over K_1. ■

Remark 3.1.14 Let (f, A) be a fuzzy soft K-subalgebra over K_1 and let (ϕ, ψ) be a fuzzy soft homomorphism from K_1 to K_2. Then $(\phi, \psi)(f, A)$ may not be a fuzzy soft K-subalgebra over K_2.

3.2 $(\in, \in \vee q)$-fuzzy soft K-algebras

Definition 3.2.1 Given a fuzzy subset μ in K and $A \subseteq [0, 1]$, we define two set-valued functions $f : A \to P(K)$ and $f_q : A \to P(K)$ by

$$f(t) = \{x \in G | x_t \in \mu\}, \quad f_q(t) = \{x \in G | x_t q \mu\}$$

for all $t \in A$, respectively. Then (f, A) and (f_q, A) are soft sets over K, which are called an $\in$-soft set and a q-soft set over K, respectively.

Example 3.2.2 Consider the K-algebra $K = (G, \cdot, \odot, e)$, where $G = \{e, a, a^2, a^3, a^4\}$ is the cyclic group of order 5 and $\odot$ is given by the following Cayley's table:

$\odot$	e	a	a^2	a^3	a^4
e	e	a^4	a^3	a^2	a
a	a	e	a^4	a^3	a^2
a^2	a^2	a	e	a^4	a^3
a^3	a^3	a^2	a	e	a^4
a^4	a^4	a^3	a^2	a	e

Let μ be a fuzzy subset in K defined by $\mu(e) = 0.7, \mu(a) = 0.8, \mu(a^2) = 0.8$

and $\mu(a^3) = \mu(a^4) = 0.4$. Then μ is an $(\in, \in \vee q)$-fuzzy K-subalgebra of K. Take $A = (0, 0.5]$ and let (f, A) be an $\in$-soft set over K. Then
(1) $f(t) = G$ if $t \in (0, 0.4]$,
(2) $f(t) = \{e, a, a^2\}$ if $t \in (0.4, 0.6]$
which are K-subalgebras of K. Hence (f, A) is a soft K-algebra over K.

Definition 3.2.3 Let (f_μ, A) be a fuzzy soft set over K. Then (f_μ, A) is said to be an $(\in, \in \vee q)$-fuzzy soft K-subalgebra over K if $f_\mu(\varepsilon)$ is an $(\in, \in \vee q)$-fuzzy K-subalgebra of K for all $\varepsilon \in A$.

Lemma 3.2.4 [2] A fuzzy subset μ in K is an $(\in, \in \vee q)$-fuzzy K-subalgebra of K if and only if it satisfies:
(i) $\mu(e) \geq \min\{\mu(x), 0.5\}$,
(ii) $\mu(x \odot y) \geq \min\{\mu(x), \mu(y), 0.5\}$
for all $x, y \in G$.

Example 3.2.5 Consider the K-algebra $K = (G, \cdot, \odot, e)$, where $G = \{e, a, a^2, a^3\}$ is the cyclic group of order 4 and $\odot$ is given by the Cayley's table in Example 3.1.3. Let $A = \{e_1, e_2\}$ and $f_\mu : A \to \tilde{P}(K)$ be a set-valued function defined by

$$f_\mu(e_1) = \{(e, f_\mu(e)), (a, f_\mu(a)), (a^2, f_\mu(a^2)), (a^3, f_\mu(a^3))\},$$

$$f_\mu(e_1) = \{(e, 0.7), (a, 0.4), (a^2, 0.6), (a^3, 0.4)\},$$

$$f_\mu(e_2) = \{(e, f_\mu(e)), (a, f_\mu(a)), (a^2, f_\mu(a^2)), (a^3, f_\mu(a^3))\},$$

$$f_\mu(e_2) = \{(e, 0.8), (a, 0.5), (a^2, 0.7), (a^3, 0.5)\}.$$

Clearly, (f_μ, A) is fuzzy soft set over K. Since $f_\mu(e) \geq \min\{f_\mu(x), 0.5\}$, $f_\mu(x \odot y) \geq \min\{f_\mu(x), f_\mu(y), 0.5\}$ hold for all $x, y \in G$, $f_\mu(\varepsilon)$ is an $(\in, \in \vee q)$-fuzzy K-subalgebra of K for all $\varepsilon \in A$. Hence (f_μ, A) is an $(\in, \in \vee q)$-fuzzy soft K-subalgebra over K.

The proofs of the following propositions are similar to the proofs of the propositions 3.1.4, 3.1.5 and 3.1.6.

Proposition 3.2.6 Let (f_μ, A) and (g_μ, B) be $(\in, \in \vee q)$-fuzzy soft K-subalgebras over K, then $(f_\mu, A)\tilde{\cap}(g_\mu, B)$ is an $(\in, \in \vee q)$-fuzzy soft K-subalgebra over K.

Proposition 3.2.7 Let (f_μ, A) and (g_μ, B) be $(\in, \in \vee q)$-fuzzy soft K-subalgebras over K, then $(f_\mu, A) \wedge (g_\mu, B)$ is an $(\in, \in \vee q)$-fuzzy soft K-subalgebra over K.

Proposition 3.2.8 Let (f_μ, A) and (g_μ, B) be $(\in, \in \vee q)$-fuzzy soft K-subalgebras over K. If $A \cap B = \emptyset$ then $(f_\mu, A)\tilde{\cup}(g_\mu, B)$ is an $(\in, \in \vee q)$-fuzzy soft K- subalgebra over K.

Proposition 3.2.9 Let μ be a fuzzy subset in K and let (f, A) be an $\in$-soft set over K with $A = (0, 1]$. Then (f, A) is a soft K-algebra over K if and only if μ is a fuzzy K-subalgebra of K.

Proof. Assume that (f, A) is a soft K-algebra over K. If μ is not a fuzzy K-subalgebra of K, then there exist $a, b \in G$ such that $\mu(a \odot b) < min\{\mu(a), \mu(b)\}$. Take $t \in A$ such that $\mu(a \odot b) < t \leq min\{\mu(a), \mu(b)\}$. Then $a_t \in \mu$ and $b_t \in \mu$ but $(a \odot b)_{\min\{t,t\}} = (a \odot b)_t \notin \mu$. Hence $a, b \in f(t)$, but $a \odot b \notin f(t)$, a contradiction. Thus, $\mu(x \odot y) \geq min\{\mu(x), \mu(y)\}$ for all $x, y \in G$. Conversely, suppose that μ is a fuzzy K-subalgebra of K. Let $t \in A$ and $x, y \in f(t)$. Then x_t and $y_t \in \mu$. It follows from Proposition 1.2.19 that $(x \odot y)_t = (x \odot y)_{\min\{t,t\}} \in \mu$ so that $x \odot y \in f(t)$.Likewise, $e \in f(t)$. Hence $f(t)$ is a K-subalgebra of K, i.e., (f, A) is a soft K-algebra over K. ■

Proposition 3.2.10 Let μ be a fuzzy subset in K and let (f_q, A) be an q-soft set over K with $A = (0, 1]$. Then (f_q, A) is a soft K-algebra over K if and only if μ is a fuzzy K-subalgebra of K

Proof. Suppose that μ is a fuzzy K-subalgebra of K. Let $t \in A$ and $x, y \in f_q(t)$. Then $x_t q\mu$ and $y_t q\mu$, i.e., $\mu(x) + t > 1$ and $\mu(y) + t > 1$. It follows from Definition 1.2.10 that

$$\mu(x \odot y) + t \geq \min\{\mu(x), \mu(y)\} + t = \min\{\mu(x) + t, \mu(y) + t\} > 1$$

so that $(x \odot y)_t q\mu$, i.e., $x \odot y \in f_q(t)$. Likewise, $e \in f_q(t)$. Hence $f_q(t)$ is a K-subalgebra of K for all $t \in A$, and so (f_q, A) is a soft K-algebra over K. The proof of converse part is obvious. ■

Proposition 3.2.11 Let μ be a fuzzy subset in K and let (f, A) be an $\in$-soft set over K with $A = (0.5, 1]$. Then the following assertions are equivalent:
(i) (f, A) is a soft K-algebra over K,
(ii) $\max\{\mu(x \odot y), 0.5\} \geq \min\{\mu(x), \mu(y)\}$.
for all $x, y \in G$.

Proof. Assume that (f, A) is a soft K-algebra over K. Then $f(t)$ is a K-subalgebra of K for all $t \in A$. If there exist $a, b \in G$ such that

$$\max\{\mu(a \odot b), 0.5\} < t = \min\{\mu(a), \mu(b)\},$$

then $t \in A, a_t \in \mu$ and $b_t \in \mu$ but $(a \odot b)_t \overline{\in} \mu$. It follows that $a, b \in f(t)$ and $a \odot b \notin f(t)$. This is a contradiction, and so

$$\max\{\mu(x \odot y), 0.5\} \geq \min\{\mu(x), \mu(y)\}$$

for all $x, y \in G$. Conversely, suppose that (ii) is valid. Let $t \in A$ and $x, y \in f(t)$. Then $x_t \in \mu$ and $y_t \in \mu$, or equivalently, $\mu(x) \geq t$ and $\mu(y) \geq t$. Hence

$$\max\{\mu(x \odot y), 0.5\} \geq \min\{\mu(x), \mu(y)\} \geq t > 0.5,$$

and thus $\mu(x \odot y) \geq t$, i.e., $(x \odot y)_t \in \mu$. Therefore $x \odot y \in f(t)$ which shows that (f, A) is a soft K-algebra over K. ■

Proposition 3.2.12 Let μ be a fuzzy subset in a K-algebra K and let (f, A) be an $\in$-soft set over K with $A = (0, 0.5]$. Then the following assertions are equivalent:
(i) μ is an $(\in, \in \vee q)$-fuzzy K-subalgebra of K,
(ii) (f, A) is a soft K-algebra over K.

Proof. Assume that μ is an $(\in, \in \vee q)$-fuzzy K-subalgebra of K. Let $t \in A$ and $x, y \in f(t)$. Then $x_t \in \mu$ and $y_t \in \mu$ or equivalently $\mu(x) \geq t$ and $\mu(y) \geq t$. It follows from Lemma 3.2.4 that

$$\mu(x \odot y) \geq \min\{\mu(x), \mu(y), 0.5\} \geq \min\{t, 0.5\} = t,$$

so that $(x \odot y)_t \in \mu$, or equivalently $x \odot y \in f(t)$. Likewise, $e \in f(t)$. Hence (f, A) is a soft K-algebra over K. Conversely, suppose that (ii) is valid. If there exist $a, b \in G$ such that

$$\mu(a \odot b) < \min\{\mu(a), \mu(b), 0.5\},$$

then we take $t \in (0, 1)$ such that $\mu(a \odot b) < t \leq \min\{\mu(a), \mu(b), 0.5\}$. Thus $t \leq 0.5$, $a_t \in \mu$ and $b_t \in \mu$, that is, $a \in f(t)$ and $b \in f(t)$. Since $f(t)$ is a K-subalgebra of K. it follows that $a \odot b \in f(t)$ for all $t \leq 0.5$ so that $(a \odot b)_t \in \mu$ or equivalently $\mu(a \odot b) \geq t$ for all $t \leq 0.5$, a contradiction. Hence

$$\mu(x \odot y) \geq \min\{\mu(x), \mu(y), 0.5\} \textit{ for all } x, y \in G.$$

Likewise, $\mu(e) \geq \min\{\mu(x), 0.5\}$ for all $x \in G$. It follows from Lemma 3.2.4 that μ is an $(\in, \in \vee q)$-fuzzy K-subalgebra of K. ■

Proposition 3.2.13 Let μ be a fuzzy subset in a K-algebra K and let (f_q, A) be a q-soft set over K with $A = (0, 1]$. Then the following assertions are equivalent:
(i) μ is a fuzzy K-subalgebra of K,
(ii) $(f_q(t) \neq \emptyset \rightarrow f_q(t))$ is a K-subalgebra of K for all $t \in A$.

Proof. Assume that μ is a fuzzy K-subalgebra of K. Let $t \in A$ be such that $f_q(t) \neq \emptyset$. Let $x, y \in f_q(t)$. Then $x_t q\mu$, $y_t q\mu$ or equivalently, $\mu(x) + t > 1$,$\mu(y) + t > 1$. Since μ is a fuzzy K-subalgebra of K, then

$$\begin{aligned}\mu(e) \geq \mu(x) &\Longrightarrow \mu(e) + t \geq \mu(x) + t > 1\\ &\Longrightarrow \mu(e) + t > 1\end{aligned}$$

i.e., $e \in f_q(t)$. Also,

$$\begin{aligned}\mu(x \odot y) \geq \min\{\mu(x), \mu(y)\} &\Longrightarrow \mu(x \odot y) \geq \mu(x) \text{ or } \mu(x \odot y) \geq \mu(y)\\ &\Longrightarrow \mu(x \odot y) + t \geq \mu(x) + t > 1 \text{ or } \mu(x \odot y) + t \geq \mu(y) + t > 1\\ &\Rightarrow \mu(x \odot y) + t > 1\end{aligned}$$

Hence $x \odot y \in f_q(t)$. Thus $f_q(t)$ is a K-subalgebra of K. Conversely, assume that (ii) is valid. Suppose there exist $a, b \in G$ such that $\mu(a \odot b) < \min\{\mu(a), \mu(b)\}$. Then $\mu(a \odot b) + s \leq 1 < \min\{\mu(a), \mu(b)\} + s$ for some $s \in A$. It follows that $(a)_s q\mu$ and $(b)_s q\mu$, i.e., $a \in f_q(s)$ and $b \in f_q(s)$. Since $f_q(s)$ is a K-subalgebra of K, we get $a \odot b \in f_q(s)$, and so $(a \odot b)_s q\mu$ or equivalently $\mu(a \odot b) + s > 1$, a contradiction. Thus $\mu(x \odot y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in G$. Hence μ is a fuzzy K-subalgebra of K. ■

3.3 $(\in_\alpha, \in_\alpha \vee q_\beta)$-fuzzy soft K-subalgebras

Let $\alpha, \beta \in [0, 1]$ be such that $\alpha < \beta$. For any $Y \subseteq X$, we define $X^\beta_{\alpha Y}$ as the fuzzy subest of X by $X^\beta_{\alpha Y}(x) \geq \beta$ for all $x \in Y$ and $X^\beta_{\alpha Y}(x) \leq \alpha$ otherwise. Clearly, $X^\beta_{\alpha Y}$ is the characteristic function of Y if $\alpha = 0$ and $\beta = 1$. For a fuzzy point x_r and a fuzzy subset μ of X, we say that

- $x_r \in_\alpha \mu$ if $\mu(x) \geq r > \alpha$
- $x_r q_\beta \mu$ if $\mu(x) + r > 2\beta$

- $x_r \in_\alpha \vee q_\beta \mu$ if $x_r \in_\alpha \mu$ or $x_r q_\beta \mu$

An ordering relation on $F(x)$, denoted as " $\subseteq q_{(\alpha,\beta)}$", is defined as follows. For any $\mu, \nu \in F(x)$, by $\mu \subseteq \vee q_{(\alpha,\beta)}\nu$ we mean that $x_r \in_\alpha \mu$ implies $x_r \in_\alpha \vee q_\beta \nu$ for all $x \in X$ and $r \in (\alpha, 1]$. Moreover, μ and ν are said to be (α,β)-equal, denoted by $\mu =_{(\alpha,\beta)} \nu$, if $\mu \subseteq \vee q_{(\alpha,\beta)}\nu$ and $\nu \subseteq \vee q_{(\alpha,\beta)}\mu$. In the sequel, unless otherwise stated, $\bar{\alpha}$ means α does not hold, where $\alpha \in \{\in_\alpha, q_\beta, \in_\alpha \vee q_\beta\}$.

We now introduce the generalized soft fuzzy K-subalgebras and describe some of their properties.

Definition 3.3.1 A fuzzy subset f_ε in a K-algebra K is called an $(\in_\alpha, \in_\alpha \vee q_\beta)$- fuzzy K-subalgebra of K if it satisfies the following conditions:

(1) $x_r \in_\alpha f_\varepsilon \rightarrow (e)_r \in_\alpha \vee q_\beta f_\varepsilon$ for all $x \in G$, $\forall\, r \in (\alpha, 1]$,

(2) $x_r, y_s \in_\alpha f_\varepsilon \rightarrow (x \odot y)_{\min\{r,s\}} \in_\alpha \vee q_\beta f_\varepsilon$ for all $x, y \in G$, $\forall\, r, s \in (\alpha, 1]$.

Definition 3.3.2 Let (f, A) be a fuzzy soft set over K. Then (f, A) is said to be an $(\in_\alpha, \in_\alpha \vee q_\beta)$-fuzzy soft K-subalgebra over K if f_ε is an $(\in_\alpha, \in_\alpha \vee q_\beta)$- fuzzy K-subalgebra of K for all $\varepsilon \in A$.

Example 3.3.3 Consider the K-algebra $K = (G, \cdot, \odot, e)$, where $G = \{e, a, a^2, a^3, a^4\}$ is the cyclic group of order 5 and $\odot$ is given by the Cayley's table of Example 3.2.2. Let $A = (0.1, 0.4]$. Define a fuzzy soft set (f, A) over K as follows.

$$f_\varepsilon(e) = 0.7, f_\varepsilon(a) = f_\varepsilon(a^2) = 0.8 \text{ and } f_\varepsilon(a^3) = f_\varepsilon(a^4) = 0.4$$

Then it is easy to verify that (f, A) is an $(\in_{0.1}, \in_{0.1} \vee q_{0.4})$-fuzzy soft K-subalgebra.

Lemma 3.3.4 Let (f, A) be an $(\in_\alpha, \in_\alpha \vee q_\beta)$-fuzzy soft K-subalgebra over K. Then

(i) $x_r \in_\alpha f_\varepsilon$ and $y_s \in_\alpha f_\varepsilon$ imply $(x \odot y)_{\min\{r,s\}} \in_\alpha \vee q_\beta f_\varepsilon$ for all $x, y \in G$,

$\varepsilon \in A$ and $r, s \in (\alpha, 1]$,

(ii) $\max\{f_\varepsilon(x \odot y), \alpha\} \geq \min\{f_\varepsilon(x), f_\varepsilon(y), \beta\}$ for all $x, y \in G$ and $\varepsilon \in A$.

Remark 3.3.5 For any fuzzy soft set (f, A) over a K-algebra K, $\varepsilon \in A$ and $r \in (\alpha, 1]$, denote $(f_\varepsilon)_r = \{x \in G : x_r \in_\alpha f_\varepsilon\}, \langle f_\varepsilon \rangle_r = \{x \in G : x_r q_\beta f_\varepsilon\}$ and $[f_\varepsilon]_r = \{x \in G : x_r \in_\alpha \vee q_\beta f_\varepsilon\}$.

Theorem 3.3.6 Let K be a K-algebra and (f, A) a fuzzy soft set over K. Then

(i) (f, A) is an $(\in_\alpha, \in_\alpha \vee q_\beta)$-fuzzy soft K-subalgebra over K if and only if nonempty subset $(f_\varepsilon)_r$ is a K-subalgebra of K for all $\varepsilon \in A$ and $r \in (\alpha, \beta]$.

(ii) If $2\beta = 1 + \alpha$, then (f, A) is an $(\in_\alpha, \in_\alpha \vee q_\beta)$-fuzzy soft K-subalgebra over K if and only if nonempty subset $\langle f_\varepsilon \rangle_r$ is a K-subalgebra of K for all $\varepsilon \in A$ and $r \in (\beta, 1]$.

(iii) (f, A) is an $(\in_\alpha, \in_\alpha \vee q_\beta)$-fuzzy soft K-subalgebra over K if and only if nonempty subset $[f_\varepsilon]_r$ is a K-subalgebra of K for all $\varepsilon \in A$ and $r \in (\alpha, \min\{2\beta - \alpha, 1\}]$.

Proof. (i) Let (f, A) be an $(\in_\alpha, \in_\alpha \vee q_\beta)$-fuzzy soft K-subalgebra over K and assume that $(f_\varepsilon)_r \neq \emptyset$ for some $\varepsilon \in A$ and $r \in (\alpha, \beta]$. Let $x, y \in (f_\varepsilon)_r$. Then $x_r \in_\alpha f_\varepsilon$ and $y_r \in_\alpha f_\varepsilon$, that is, $f_\varepsilon(x) \geq r > \alpha$ and $f_\varepsilon(y) \geq r > \alpha$. Since (f, A) is an $(\in_\alpha, \in_\alpha \vee q_\beta)$-fuzzy soft K-subalgebra over K, we have $\max\{f_\varepsilon(x \odot y), \alpha\} \geq \min\{f_\varepsilon(x), f_\varepsilon(y), \beta\}$ and so $\max\{f_\varepsilon(x \odot y), \alpha\} \geq r > \alpha$. Hence $f_\varepsilon(x \odot y) \geq r > \alpha$, i.e., $x \odot y \in (f_\varepsilon)_r$. Therefore, $(f_\varepsilon)_r$ is a K-subalgebra of K. Conversely, assume that the given conditions hold. If there exist $\varepsilon \in A$ and $x, y \in G$ such that $\max\{f_\varepsilon(x \odot y), \alpha\} < r = \min\{f_\varepsilon(x), f_\varepsilon(y), \beta\}$. Then $x_r \in_\alpha f_\varepsilon$, $y_r \in_\alpha f_\varepsilon$ but $(x \odot y)_r \overline{\in_\alpha} f_\varepsilon$, that is, $x \in (f_\varepsilon)_r$, $y \in (f_\varepsilon)_r$ but $x \odot y \notin (f_\varepsilon)_r$, a contradiction. Therefore, (f, A) is an $(\in_\alpha, \in_\alpha \vee q_\beta)$-fuzzy soft K-subalgebra over K.

(ii) Assume that $2\beta = 1 + \alpha$. Let (f, A) be an $(\in_\alpha, \in_\alpha \vee q_\beta)$-fuzzy soft K-subalgebra over K and assume that $\langle f_\varepsilon \rangle_r \neq \emptyset$ for some $\varepsilon \in A$ and $r \in$

$(\beta, 1]$. Let $x, y \in \langle f_\varepsilon \rangle_r$. Then $x_r q_\beta f_\varepsilon$ and $y_r q_\beta f_\varepsilon$, that is, $f_\varepsilon(x) + r > 2\beta$ and $f_\varepsilon(y) + r > 2\beta$. Since (f, A) is an $(\in_\alpha, \in_\alpha \vee q_\beta)$-fuzzy soft K-subalgebra over K, we have $\max\{f_\varepsilon(x \odot y), \alpha\} \geq \min\{f_\varepsilon(x), f_\varepsilon(y), \beta\}$. Thus by $r > \beta$,

$$\begin{aligned}
\max\{f_\varepsilon(x \odot y) + r, \alpha + r\} &= \max\{f_\varepsilon(x \odot y), \alpha\} + r \\
&\geq \min\{f_\varepsilon(x), f_\varepsilon(y), \beta\} + r \\
&= \min\{f_\varepsilon(x) + r, f_\varepsilon(y) + r, \beta + r\} \\
&> 2\beta
\end{aligned}$$

From $r \leq 1 = 2\beta - \alpha$, that is, $r + \alpha \leq 2\beta$, we have $f_\varepsilon(x \odot y) + r > 2\beta$ and so $x \odot y \in \langle f_\varepsilon \rangle_r$. Likewise, $e \in \langle f_\varepsilon \rangle_r$ Therefore, $\langle f_\varepsilon \rangle_r$ is a K-subalgebra of K. Conversely, assume that the given conditions hold. If there exist $\varepsilon \in A$ and $x, y \in G$ such that $\max\{f_\varepsilon(x \odot y), \alpha\} < \min\{f_\varepsilon(x), f_\varepsilon(y), \beta\}$. Take $r = 2\beta - \max\{f_\varepsilon(x \odot y), \alpha\}$. Then $r \in (\beta, 1]$, $f_\varepsilon(x \odot y) \leq 2\beta - r$, $f_\varepsilon(x)$ $> \max\{f_\varepsilon(x \odot y), \alpha\} = 2\beta - r$, $f_\varepsilon(y) > \max\{f_\varepsilon(x \odot y), \alpha\} = 2\beta - r$, that is, $x \in \langle f_\varepsilon \rangle_r$, $y \in \langle f_\varepsilon \rangle_r$ but $x \odot y \notin \langle f_\varepsilon \rangle_r$, a contradiction. Therefore, (f, A) is an $(\in_\alpha, \in_\alpha \vee q_\beta)$-fuzzy soft K-subalgebra over K.

(iii) Let (f, A) be an $(\in_\alpha, \in_\alpha \vee q_\beta)$-fuzzy soft K-subalgebra over K and assume that $[f_\varepsilon]_r \neq \emptyset$ for some $\varepsilon \in A$ and $r \in (\alpha, \min\{2\beta - \alpha, 1\}]$. Let $x, y \in [f_\varepsilon]_r$. Then $x_r \in_\alpha \vee q_\beta f_\varepsilon$ and $y_r \in_\alpha \vee q_\beta f_\varepsilon$, that is, $f_\varepsilon(x) \geq r > \alpha$ or $f_\varepsilon(x) > 2\beta - r \geq 2\beta - (2\beta - \alpha) = \alpha$ and $f_\varepsilon(y) \geq r > \alpha$ or
$f_\varepsilon(y) > 2\beta - r \geq 2\beta - (2\beta - \alpha) = \alpha$. Since (f, A) is an $(\in_\alpha, \in_\alpha \vee q_\beta)$-fuzzy soft K-subalgebra over K, we have $\max\{f_\varepsilon(x \odot y), \alpha\} \geq \min\{f_\varepsilon(x), f_\varepsilon(y), \beta\}$ and so $f_\varepsilon(x \odot y) \geq \min\{f_\varepsilon(x), f_\varepsilon(y), \beta\}$ since $\alpha < \min\{f_\varepsilon(x), f_\varepsilon(y), \beta\}$ in any case.

We now consider the following cases.

Case 1: $r \in (\alpha, \beta]$. Then $2\beta - r \geq \beta \geq r$. If $f_\varepsilon(x) \geq r$ and $f_\varepsilon(y) \geq r$ or $f_\varepsilon(x) > 2\beta - r$ and $f_\varepsilon(y) > 2\beta - r$, then $f_\varepsilon(x \odot y) \geq \min\{f_\varepsilon(x), f_\varepsilon(y), \beta\} \geq r$. Hence $(x \odot y)_r \in_\alpha f_\varepsilon$.

Case 2: $r \in (\beta, \min\{2\beta - \alpha, 1\}]$. Then $r > \beta > 2\beta - r$. If $f_\varepsilon(x) \geq r$ and $f_\varepsilon(y) \geq r$ or $f_\varepsilon(x) > 2\beta - r$ and $f_\varepsilon(y) > 2\beta - r$, then

$f_\varepsilon(x \odot y) \geq \min\{f_\varepsilon(x), f_\varepsilon(y), \beta\} > 2\beta - r$. Hence $(x \odot y)_r q_\beta f_\varepsilon$. Thus, in any case, $(x \odot y)_r \in_\alpha \vee q_\beta f_\varepsilon$, that is, $x \odot y \in [f_\varepsilon]_r$. Therefore, $[f_\varepsilon]_r$ is a K-subalgebra. Conversely, assume that the given conditions hold. If there exist $\varepsilon \in A$ and $x, y \in G$ such that
$\max\{f_\varepsilon(x \odot y), \alpha\} < r = \min\{f_\varepsilon(x), f_\varepsilon(y), \beta\}$. Then $x_r \in_\alpha f_\varepsilon$, $y_r \in_\alpha f_\varepsilon$ but $(x \odot y)_r \overline{\in_\alpha \vee q_\beta} f_\varepsilon$, that is, $x \in [f_\varepsilon]_r$, $y \in [f_\varepsilon]_r$ but $x \odot y \notin [f_\varepsilon]_r$, a contradiction. Therefore, (f, A) is an $(\in_\alpha, \in_\alpha \vee q_\beta)$-fuzzy soft K-subalgebra over K. ■

3.4 conclusions

Presently, science and technology is featured with complex processes and phenomena for which complete information is not always available. For such cases, mathematical models are developed to handle various types of systems containing elements of uncertainty. A large number of these models is based on an extension of the ordinary set theory, namely, fuzzy sets and soft sets. The fuzzy sets and soft sets are two different methods for representing uncertainty, we have applied these methods in combination to study uncertainty in K-subalgebras in this thesise. A K-algebra is relatively a new branch of logical algebras and there are many unsolved problems in it. In our opinion the future study of K-algebras can be extended with the following: (i) Hyper K-algebras, (ii) Roughness in K-algebras, and (iii) Automorphic K-algebras.

Bibliography

[1] U. Acar, F. Koyuncu and B. Tanay, Soft sets and soft rings, Computers and Mathematics with Applications 59 (2010) 3458-3463.

[2] M. Akram, $(\in, \in \vee q)$-fuzzy ideals of K-algebras, Ars Combinatoria 89(2008) 191-204.

[3] M. Akram, Bifuzzy K-algebras, VDM Verlag, 2010, pp.142, ISBN 978-3-639-28648-9

[4] M. Akram and N. O. Al-Shehri, Fuzzy K-ideals of K-algebras, Ars Combinatoria 99(2011) 399-413.

[5] M. Akram and K. H. Dar, Fuzzy ideals of K-algebras, Annals of University of Craiova, Math. Comp. Sci. Ser 34(2007) 3-12.

[6] M. Akram and K. H. Dar, Generalized fuzzy K-algebras, VDM Verlag, 2010.

[7] M. Akram and H. S. Kim, On K-algebras and BCI-algebras, International Mathematical Forum 2 (2007), no. 9-12, 583–587.

[8] M. Akram, N. O. Al-Shehri and R. S. Alghamdi, Fuzzy soft K-algebras, Utilitas Mathematica 90(2013) pp. 19.

[9] M. Akram, K. H. Dar and K.P. Shum, Interval-valued (α, β)- fuzzy K-algebras, Applied Soft computing, 11(1)(2011) 1213-1222.

[10] M. Akram, K. H. Dar, Y. B. Jun and E. H. Roh, Fuzzy structures on $K(G)$-algebra, Southeast Asian Bulletin of Mathematics 31(4)(2007) 625-637.

[11] M. Akram, K. H. Dar, B.L. Meng and Y.L. Liu, Redefined fuzzy K-algebras, World Applied Sciences Journal 7(2009)805-811.

[12] H. Aktas and N. Cagman, Soft sets and soft groups, Information Sciences 177(13)(2007) 2726-2735.

[13] M. I. Ali, A note on soft sets, rough soft sets and fuzzy soft sets, Applied Soft Computing 11 (2011) 3329–3332.

[14] M. I. Ali and M. Shabir, Soft ideals and generalized fuzzy ideals in semigroups, New Math. and Nat. Comp., 5(2009) 599-615.

[15] M. I. Ali, F. Feng, X. Y. Liu, W. K. Min and M. Shabir, On some new operations in soft set theory, Comput. Math. Appl. 57(2009) 1547-1553.

[16] N. O. Al-Shehri, M. Akram and R. S. Alghamdi, Applications of soft sets in K-algebras, Advances in Fuzzy Systems, Volume 2013, Article ID 319542,8pages

[17] A Ayg¨unoglu and H. Ayg¨un, Introduction to fuzzy soft groups, Comput. Math. Appl. 58(6)(2009) 1279-1286.

[18] T. M. Basu, N. K. Mahapatra and S. K. Mondal, On some New Operations in Fuzzy Soft Set And Intuitionistic Fuzzy Soft Set Theory, International Journal of Computer Science and Information Technology & Security, 2(2)(2012) 2249-9555.

[19] N. Ca gman, F. Citak and K. Aktas, Soft int-groups and its applications to group theory, Neural Comput Appl. doi:10.1007/s00521-011-0752-x, 2012.

[20] D. Chen, E.C.C. Tsang, D. S. Yeung and X. Wang, The parameterization reduction of soft sets and its applications, Computers and Mathematics with Applications 49(5-6)(2005)757-763.

[21] K. H. Dar and M. Akram, Characterization of a $K(G)$-algebra by self maps, Southeast Asian Bulletin of Mathematics 28 (2004) 601-610.

[22] K. H. Dar and M. Akram, On a K-algebra built on a group, Southeast Asian Bulletin of Mathematics 29(1)(2005) 41-49.

[23] K.H. Dar and M. Akram, On subclasses of $K(G)$-algebras, Annals of University of Craiova, Math. Comp. Sci. Ser 33(2006) 235-240.

[24] K.H. Dar and M. Akram, On K-homomorphisms of K-algebras, International Mathematical Forum 46(2007) 2283-2293.

[25] K. H. Dar and M. Akram, Characterization of K-algebras by self maps II, Annals of University of Craiova, Math. Comp. Sci. Ser. 37(1)(2010)96-103.

[26] F. Feng, Y. B. Jun and X. Z. Zhao, Soft semirings, Comput. Math. Appl. 56 (2008) 2621-2628.

[27] F. Feng, Y. B. Jun, X.Y. Liu and L.F. Li, An adjustable approach to fuzzy soft set based decision making, Journal of Computational and Applied Mathematics 234 (2010)10-20.

[28] F. Feng, C. X. Li, B. Davvaz, M. Irfan Ali, Soft sets combined with fuzzy sets and rough sets: a tentative approach, Soft Computing 14 (2010) 899–911.

[29] F. Feng, X.Y. Liu, V. Leoreanu-Fotea, Y.B. Jun, Soft sets and soft rough sets, Information Sciences 181 (2011) 1125-1137.

[30] A. H. Handam, soft $K(G)$-algebras, Tamkang Journal of Mathematics 43(2)(2012)203-213.

[31] Y. Imai and K. Iseki, On axiom system of propositional calculi XIV, Proc. Japonica Acad 42(1966)19-22.

[32] K. Iseki, An algebra related with a propositional calculus, Proc. Japan Acad Vol. 42(1966)26-29.

[33] Y .B. Jun and C.H. Park, Fuzzy idelas of $K(G)$-algebras, Honam Mathematical Journal 28(2006) 485-497.

[34] Y. B. Jun, K.J. Lee and E.H. Roh, Intersectional soft BCK/BCI-ideals, Ann. Fuzzy Math. Inform. 4(1) (2012) 1-7.

[35] O. Kazanci, S. Ylmaz and S. Yama, Soft sets and soft BCH-algebras, Hacettepe Journal of Mathematics and Statistics 39 (2) (2010) 205-217.

[36] P. K. Maji, R. Biswas and R.Roy, Fuzzy soft sets, The J. of Fuzzy Math. 9(3)(2001) 589-602.

[37] P. K. Maji, R. Biswas and R. Roy, Soft set theory, Comput. Math. Appl. 45(2003) 555-562.

[38] P. K. Maji, A. R. Roy and R. Biswas, An application of soft sets in a decision making problem, Comput. Math. Appl. 44 (2002) 1077-1083.

[39] S. V. Manemaran, On fuzzy soft groups, International Journal of Computer Application 15(7)(2011) 38-44.

[40] D. Molodtsov, Soft set theory-first results, Comput. Math. Appl. 37(1999) 19-31.

[41] J. Neggers and H. S. Kim, On B-algebras, Matematicki Vesnik 54 (2002)21-29.

[42] P. M. Pu and Y. M. Liu, Fuzzy topology,I. Neighborhood structure of a fuzzy point and Moore-Smith convergence , J. Math. Anal. Appl. 76(2) (1980) 571-599.

[43] A. R. Roy and P. K. Maji, A fuzzy soft set theoretic approach to decision making problems, J. Comput. Appl. Math. 203 (2007) 412-418.

[44] A. Sezgin and A. O. Atagun, Soft groups and normalistic soft groups, Computers and Mathematics with Applications 62 (2011) 685-698.

[45] A. Sezgin, A. O. Atagun and E. Aygun, A note on soft near-rings and idealistic soft near-rings, Filomat 25(2)(2011) 53-68.

[46] C.-F. Yang, Fuzzy soft semigroups and fuzzy soft ideals,Comput. Math. Appl. 61(2011) 255-561.

[47] L.A. Zadeh, Fuzzy sets, Inform. and Control 8 (1965) 338-353.

[48] L.A. Zadeh, The concept of a linguistic and application to approximate reasoning I, Information Sci. 8 (1975) 199-249.

[49] J. Zhan and Y. Xu, Soft lattice implication algebras based on fuzzy sets, Hacettepe Journal of Mathematics and Statistics 40(2011) 483-492.

[50] J. Zhan and W. A. Dudek, Soft MTL-algebras based on fuzzy sets, UPB Scienti c Bulletin, Series A: Applied Mathematics and Physics 74(2)(2012)40-56.